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GENERALIZED ABEL TRANSFORMATION ON SOME SEMI-DIRECT PRODUCT GROUP

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Abstract. Let G be a connected Lie group, K a compact subgroup of the group of automorphisms of the connected Lie group and δ a unitary class of irreducible representations of K. In this work, we introduce a generalized Abel transformation according to δ and study its spherical Grassmannian.

Key Words and Phrases: reductive Lie group, spherical function, spherical Fourier transform of type delta.

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1. Introduction

The Abel transformation, named after the Norwegian mathematician Niels Henrik Abel (1802-1829), finds its roots in Abel's work on integral equations in the early 19th century. This Abel transformation is a special case of the Radon transformation. It was initially developed in the context of differential and integral equations; it was extended in the 20th century to fields such as harmonic analysis, differential geometry, spectral theory, and physics. Harish-Chandra in his papers [2] and [3] extended the notion of the Abel transformation on semisimple Lie groups (1950-1960). Since the 1980s, the Abel transformation has been integrated into Harmonic analysis with applications in representation theory. It is applied in the field of signal processing in astrophysics, where it is used to analyze emission profiles of celestial objects with spherical or cylindrical symmetry. Integrated signals along lines of sight are transformed to reconstruct the spatial or spectral distribution. It is also seen as a topological isomorphism for certain function spaces.
G. Warner (See [14]) generalized the notion of the Abel transformation on semisimple Lie groups of type δ in order to determine the spherical function of type δ where δ is not necessarily trivial and the spherical Fourier transform of type δ. Thanks to this work, K.Kangni and S.Touré have given some applications on the generalized Abel transformation on semisimple Lie groups (see [6], [7] and [8]).
The goal of this paper is to construct the Abel transformation of type δ on $\tilde{G}$ when $\tilde{G}$ is a Semi-direct product group of a connected Lie group G and a compact subgroup K of the group of automorphisms of G, and to define an extension of this Abel transformation on reductive Lie groups.
In Section 2, we construct the Abel transformation of type δ on $\tilde{G} = K \rtimes G$ and we give an application of the Abel transformation to groups $\tilde{G} = SU(2) \rtimes H_2$ with $K = SU(2)$.
Finally, we generalize the Abel transformation on reductive Lie groups.

2. Generalized Abel Transformation on semi-direct product group

Let G be a connected Lie group and K be a compact subgroup of $\mathrm{Aut}(G)$, the group of automorphisms of G.

The action of K on G is defined by:

$$\begin{aligned}\alpha : K \times G &\longrightarrow G \\ (k, \tilde{g}) &\longmapsto k \cdot \tilde{g} = k(\tilde{g})\end{aligned}$$

We set $\tilde{G} := K \rtimes G$, the semi-direct product of K and G with the group law. For all $(k_1, g_1), (k_2, g_2) \in \tilde{G}$, we put

$$\begin{aligned}(k_1, g_1)(k_2, g_2) &= (k_1k_2, g_1\alpha_{k_1}(g_2)) \\ &= (k_1k_2, g_1k_1 \cdot g_2).\end{aligned}$$

Let $\hat{K}$ denote the set of all equivalence classes of finite dimensional irreducible representations of K. For any class δ of $\hat{K}$, let ζ_δ be the character of δ, $d(\delta)$ be the degree of δ and $\chi_\delta = d(\delta)\zeta_\delta$.

Thanks to the orthogonality relations of Schur, we easily check that $\chi_\delta * \chi_\delta = \chi_\delta$. We denote by $M_{d(\delta)}(\mathbb{C})$, the algebra of square matrices of order $d(\delta)$ with complex coefficients. Let us consider $I_c(\tilde{G})$, the set of all K-central functions. $f \in I_c(\tilde{G}) \iff f : \tilde{G} \to \mathbb{C}$ is a continuous function with compact support and

$$f(k\tilde{k}k^{-1}, k \cdot \tilde{g}) = f(\tilde{k}, \tilde{g}) \quad \forall\, \tilde{g} \in G, \forall\, k, \tilde{k} \in K.$$

By identifying χ_δ with a bounded measure on $\tilde{G}$, we set for any function $f \in I_{c;\delta}(\tilde{G})$:

$$\chi_\delta * f(\tilde{k}, \tilde{g}) = \int_K \chi_\delta(k^{-1})\, f(k\tilde{k}, k.\tilde{g})\, dk.$$

$$f * \chi_\delta(\tilde{k}, \tilde{g}) = \int_K \chi_\delta(k^{-1})\, f(k\tilde{k}, \tilde{g})\, dk.$$

Let $I_\delta(\tilde{G})$ be the set of all continuous, complex-valued function f on $\tilde{G}$ with compact support such that:

$$\chi_\delta * f = f * \chi_\delta = f.$$

Let us put $I_{c,\delta}(\tilde{G}) = I_c(\tilde{G}) \cap I_\delta(\tilde{G})$. $I_{c,\delta}(\tilde{G})$ is a subalgebra of $I(\tilde{G})$, where $I(\tilde{G})$ is the convolution algebra of all continuous, complex-valued function on $\tilde{G}$ with compact support.

For all $f \in I(\tilde{G})$, we put:

$$f_K(\tilde{k}, \tilde{g}) = \int_K f(k\tilde{k}k^{-1}, k \cdot \tilde{g})dk \ \text{ with } \tilde{k} \in K \text{ and } \tilde{g} \in G.$$

Let E be a finite-dimensional complex vector space. A spherical function Φ (on $\tilde{G}$) of type δ is a quasi-bounded continuous function on $\tilde{G}$ with values in $\mathrm{End}_{\mathbb{C}}(E)$ such that:

(i) $\Phi(k(\tilde{k}, \tilde{g})k^{-1}) = \Phi(\tilde{k}, \tilde{g})$ for all $\tilde{g} \in G, k \in K$

(ii) $\chi_\delta * \Phi = \Phi = \Phi * \chi_\delta$

(iii) The map $u_\Phi : f \mapsto \int_G f(x)\Phi(x)\, d_{\tilde{G}}(x)$ is an irreducible representation of the algebra $I_{c,\delta}(\tilde{G})$.
The dimension of E is called the height of Φ. If U is an irreducible Banach representation of G over a space E such that δ occurs m times in the restriction of U to K, then there

exists a function ϕ_δ^U defined on G which is spherical of type δ. The function ϕ_δ^U is said to be associated to the representation U.

Let $\mathcal{B}$ be a commutative, involutive Banach algebra with identity element e, and $X_m(\mathcal{B})$ the set of all m-dimensional irreducible unitary representations of $\mathcal{B}$.

For all f in $\mathcal{B}$, a generalized Gelfand transform of f is a map $\mathcal{G}f$ of $X_m(\mathcal{B})$ onto the algebra $M_m(\mathbb{C})$ of square matrices of order m defined by:

$$\mathfrak{g}(f) = u(f), \,\forall\, u \in X_m(\mathcal{B}).$$

The homomorphism $f \longmapsto \mathfrak{g}f$ of $\mathcal{B}$ onto $M_m(\mathbb{C})^{X_m(\mathcal{B})}$ is called the generalized Gelfand transformation associated to $\mathcal{B}$. Since $\mathcal{B}$ is commutative, then the irreducible unitary representations of $\mathcal{B}$ are one dimensional, hence we identify them with characters of $\mathcal{B}$. We get the usual definition of Gelfand transformation.

Let $\mathcal{G}_{m,\delta}(\tilde{G})$ be the set of all spherical functions of type δ on $\tilde{G}$ and height m. If ϕ is a function of $\mathcal{G}_{m,\delta}(\tilde{G})$, then there exists a representation $u_\delta^\phi \in X_m(I_{c,\delta}(\tilde{G}))$ such that $u_\delta^\phi(f) = \int_{\tilde{G}} f(x)\phi(x)\, d_{\tilde{G}}(x)$ and conversely. This result allows us to identify $\mathcal{G}_{m,\delta}(\tilde{G})$ with $X_m(I_{c,\delta}(\tilde{G}))$ and then, we can define $\mathcal{F}f$, the spherical Fourier transform of type δ of any $f \in I_{c,\delta}(\tilde{G})$ by:

$$\mathcal{F}f(\phi) = \int_{\tilde{G}} f(x)\,\phi(x^{-1})\,dx, \;\forall \phi \in g_{m,\delta}(\tilde{G}).$$

Let $\check{\delta}$ be a contragredient representation of δ and $u_{\check{\delta}}$ be arbitrary element in the equivalence class of $\check{\delta}$. We put $F_{\check{\delta}} = \mathrm{Hom}(E_{\check{\delta}}, E_{\check{\delta}})$ where $E_{\check{\delta}}$ is the space of the representation $u_{\check{\delta}}$.

Let us denote by $\mathscr{C}_c(\tilde{G}, F_{\check{\delta}})$, the convolution algebra of continuous functions on $\tilde{G}$, with compact support and with values in $F_{\check{\delta}}$. We set:

$$U_{c,\delta}(\tilde{G}) = \{\phi \in \mathscr{C}_c(\tilde{G}, F_{\check{\delta}}) : \phi(k_1, (\tilde{k}, \tilde{g})k_2) = u_{\check{\delta}}(k_1)\phi(\tilde{k}, \tilde{g})u_{\check{\delta}}(k_2)\}.$$

$I_{c,\delta}(\tilde{G})$ is isomorphic to $U_{c,\delta}(\tilde{G})$, thanks to the map:

$$\begin{aligned}\Psi^\delta : I_{c,\delta}(\tilde{G}) &\longrightarrow U_{c,\delta}(\tilde{G})\\ f &\longmapsto \Psi_f^\delta\end{aligned}$$

where

$$\Psi_f^\delta(\tilde{k}, \tilde{g}) = \int_K u_{\check{\delta}}(k^{-1})\, f(k\cdot(\tilde{k}, \tilde{g}))dk.$$

Let us denote by $\mathscr{C}_c(G, F_{\check{\delta}})$, the convolution algebra of continuous functions on G, with compact support and with values in $F_{\check{\delta}}$.

We set :

$$U_{c,\delta}(G) = \{\phi \in \mathscr{C}_c(G, F_{\check{\delta}}) : \phi(k\cdot\tilde{g}) = u_{\check{\delta}}(k)\,\phi(\tilde{g})u_{\check{\delta}}(k^{-1})\}.$$

Definition 2.1. Let $\delta \in \hat{K}$ and $u_{\check{\delta}} \in \check{\delta}$.
The Abel transformation of type δ : $f \longmapsto \theta_f^\delta$ is a linear mapping from $I_{c,\delta}(K \rtimes G)$ into $I_{c,\delta}(G)$ where θ_f^δ defined by:

$$\text{for all } \tilde{g} \in G,\; \theta_f^\delta(\tilde{g}) = \int_K u_{\check{\delta}}(k^{-1})\, f(k, k\cdot\tilde{g})\, dk.$$

θ_f^δ is called the Abel transform of f.

Remark 2.2. Let $\delta \in \hat{K}$ then $I_{c,\delta}(K \rtimes G)$ is isomorphic to $U_{c,\delta}(G)$.
Indeed, we can identify $I_{c,\delta}(K \rtimes G)$ with $I_{c,\delta}(G)$. Moreover, $I_{c,\delta}(G)$ is isomorphic to $U_{c,\delta}(G)$, so $I_{c,\delta}(K \rtimes G)$ is isomorphic to $U_{c,\delta}(G)$.

Theorem 2.3. *Let $\theta : f \longmapsto \theta^\delta_f$ be an Abel transformation of type δ then the following statements hold:*

(i) θ is bijective

(ii) for all $f, g \in I_{c,\delta}(K \rtimes G)$, $\theta^\delta_{f*g} = \theta^\delta_f * \theta^\delta_g$.

Proof.

i) Let $f \in I_{c,\delta}(K \rtimes G)$ then $\theta^\delta_f = \Psi_{f|G}$. By definition we have:

$$\begin{aligned}\theta^\delta_f(\tilde{g}) &:= \Psi^\delta_f(e_K, \tilde{g}) \\ &= \int_K u_{\check{\delta}}(k^{-1}) f(k, k\cdot\tilde{g})\, dk\end{aligned}$$

$$\begin{aligned}\Psi^\delta_f(\tilde{k}, \tilde{g}) &= \int_K u_{\check{\delta}}(k^{-1}) f(k\cdot(\tilde{k}, \tilde{g}))\, dk \\ &= \int_K u_{\check{\delta}}(k^{-1})\, f(k\tilde{k}, k\cdot\tilde{g})\, dk \\ &= \Psi^\delta_f((e_K, \tilde{g})(\tilde{k}, e_G)) \\ &= \Psi^\delta_f((e_K, \tilde{g})(e_K, e_G)\tilde{k}) \\ &= \Psi^\delta_f((e_K, \tilde{g})\tilde{k}) \\ &= \Psi^\delta_f(e_K, \tilde{g})\, u_{\check{\delta}}(\tilde{k}) \\ \Psi^\delta_f(\tilde{k}, \tilde{g}) &= \theta^\delta_f(\tilde{g}) u_{\check{\delta}}(\tilde{k}).\end{aligned}$$

We prove now that $\theta^\delta_f \in U_{c,\delta}(G)$, that means $\theta^\delta_f(k\cdot\tilde{g}) = u_{\check{\delta}}(k)\, \theta^\delta_f(\tilde{g}) u_{\check{\delta}}(k^{-1})$. We have,

$$\begin{aligned}\theta^\delta_f(\tilde{g}) &= \Psi^\delta_f(e_K, k\cdot\tilde{g}) \\ &= \Psi^\delta_f((k, e_G)(e_K, \tilde{g})(k^{-1}, e_G)) \\ &= \Psi^\delta_f(k(e_K, e_G)(e_K, \tilde{g})(e_K, e_G)k^{-1}) \\ &= u_{\check{\delta}}(k)\Psi^\delta_f(e_K, \tilde{g}) u_{\check{\delta}}(k^{-1}).\end{aligned}$$

So $\theta^\delta_f \in U_{c,\delta}(G)$ for all $f \in I_{c,\delta}(\tilde{G})$. Let us consider the map :

$$\begin{aligned}\tilde{\beta} : I_{c,\delta}(K \rtimes G) &\longrightarrow U_{c,\delta}(G) \\ f &\longmapsto \tilde{\beta}(f) = \theta^\delta_f.\end{aligned}$$

The linearity of $\tilde{\beta}$ is obvious. Let us show that $\tilde{\beta}$ is injective.

Let $f, g \in I_{c,\delta}(\tilde{G})$ such that $\tilde{\beta}(f) = \tilde{\beta}(g)$.

$$\begin{aligned}\tilde{\beta}(f) = \tilde{\beta}(g) &\iff \theta^\delta_f = \theta^\delta_g \\ &\iff \theta^\delta_f(\tilde{g}) = \theta^\delta_g(\tilde{g}) \\ &\iff \theta^\delta_f(\tilde{g}) u_{\check{\delta}}(\tilde{k}) = \theta^\delta_g(\tilde{g}) u_{\check{\delta}}(\tilde{k}) \quad (\tilde{k} \in K) \\ &\iff \Psi^\delta_f(\tilde{k}, \tilde{g}) = \Psi^\delta_g(\tilde{k}, \tilde{g})\end{aligned}$$

since $f \mapsto \Psi_f^\delta$ is injective, then $\tilde{\beta}(f) = \tilde{\beta}(g) \implies f = g$. So $\tilde{\beta}$ is injective.
We prove that $\tilde{\beta}$ is surjective. Let $\phi \in U_{c,\delta}(G)$ and let us put $\Psi(k, \tilde{g}) = \phi(\tilde{g}) u_{\check{\delta}}(k)$ for all $\tilde{g} \in G, k \in K$, where $\Psi \in \mathscr{C}_c(\tilde{G}, F_{\check{\delta}})$. We prove now that $\Psi \in U_{c,\delta}(\tilde{G})$

$$\begin{aligned}
\Psi(k_1(k,\tilde{g})k_2) &= \Psi((k_1, e_G),(k,\tilde{g})(k_2,e_G)) \\
&= \Psi(k_1kk_2, k\cdot\tilde{g}) \\
&= \Psi(k_1kk_2, k\cdot\tilde{g}) \\
&= \phi(k_1\cdot\tilde{g})\, u_{\check{\delta}}(k_1kk_2) \\
&= u_{\check{\delta}}(k_1)\,\phi(\tilde{g})\, u_{\check{\delta}}(k_1^{-1})u_{\check{\delta}}(k_1)u_{\check{\delta}}(k)u_{\check{\delta}}(k_2) \\
&= u_{\check{\delta}}(k_1)\,\phi(\tilde{g})\, u_{\check{\delta}}(k)u_{\check{\delta}}(k_2) \\
&= u_{\check{\delta}}(k_1)\,\Psi(k,\tilde{g})u_{\check{\delta}}(k_2).
\end{aligned}$$

Thus, $\Psi \in U_{c,\delta}(\tilde{G})$. Since $U_{c,\delta}(\tilde{G})$ is isomorphic to $I_{c,\delta}(\tilde{G})$, then $\Psi \in I_{c,\delta}(\tilde{G})$. For that reason, $\tilde{\beta}$ is surjective.

(ii) We prove now that $\tilde{\beta}$ is a convolution algebra morphism. For f,$g \in I_{c,\delta}(\tilde{G})$; we have

$$\begin{aligned}
\theta^\delta_{f*g}(x) &= \int_K u_{\check{\delta}}(k^{-1}) f*g(k, k\cdot x)dk \\
&= \int_K u_{\check{\delta}}(k^{-1}) \left(\int_K\int_G f((k,k\cdot x)(\tilde{k},y))\, g(\tilde{k}^{-1},\tilde{k}^{-1}\cdot y^{-1}\, d\tilde{k}\, dy\, dk\right) \\
&= \int_K u_{\check{\delta}}(k^{-1}) \left(\int_K\int_G f((k,k\cdot x)(\tilde{k},y))\, g(\tilde{k}^{-1},\tilde{k}^{-1}\cdot y^{-1}\, d\tilde{k}\, dy\right) \\
&= \int_K\int_K\int_G u_{\check{\delta}}(k^{-1}) f(k\tilde{k}, k\cdot(xy))\, g(\tilde{k}^{-1},\tilde{k}^{-1}\cdot y^{-1})\, dk\, d\tilde{k}\, dy \\
&= \int_K\int_G \left(\int_K u_{\check{\delta}}(k^{-1}) f(k\tilde{k}, k\cdot(xy))dk\right) g((\tilde{k}^{-1},\tilde{k}^{-1}\cdot y^{-1}))\, d\tilde{k}\, dy \\
&= \int_K\int_G \Psi^\delta_f(\tilde{k}, xy)\, g(\tilde{k}^{-1},\tilde{k}^{-1}\cdot y^{-1})\, d\tilde{k}\, dy \\
&= \int_K\int_G \Psi^\delta_f((e_k, xy)(\tilde{k}, e_G))\, g(\tilde{k}^{-1},\tilde{k}^{-1}\cdot y^{-1})\, d\tilde{k}\, dy \\
&= \int_G \theta^\delta_f(xy) \left(\int_K u_{\check{\delta}}(\tilde{k})\, g(\tilde{k}^{-1},\tilde{k}^{-1}\cdot y^{-1})\, d\tilde{k}\right) dy \\
&= \int_G \theta^\delta_f(xy) (\int_K u_{\check{\delta}}(\tilde{k}^{-1})\, g(\tilde{k},\tilde{k}\cdot y^{-1}) d\tilde{k})\, dy \\
&= \int_G \theta^\delta_f(xy)\, \theta^\delta_g(y^{-1})\, dy \\
&= \theta^\delta_f * \theta^\delta_g(x).
\end{aligned}$$

□

Remark 2.4. Let $u_{\check{\delta}}$ an irreducible unitary representation of K on $E_{\check{\delta}}$.
For an arbitrary endomorphism T of $E_{\check{\delta}}$, let us define σ by: $\sigma(T) = d(\check{\delta})\, tr(T)$. Then, thanks to the Schur orthogonality relations, it follows that

$$T = \int_K u_{\check{\delta}}(k^{-1})\, \sigma(u_{\check{\delta}}(k)T)\, dk.$$

Let ρ be a semi-norm on $K \rtimes G$, let $\hat{z}$ be a character on G and $u_{\check{\delta}}$ be an element in the class of $\check{\delta}$. Suppose that there exists $M{>}0$ such that

$$|\langle \hat{z}, \tilde{g}\rangle| \le M\rho(\tilde{g}) \text{ for all } \tilde{g} \in G.$$

Let us put

$$\varphi_\delta(f) = \int_G \langle \hat{z}, \tilde{g}\rangle \, \theta_f^\delta(\tilde{g})\, d\tilde{g} \text{ with } f \in I_{c,\delta}(\tilde{G}).$$

Theorem 2.5. *Let $\delta \in \hat{K}$, the map φ_δ defined from $I_{c,\delta}(\tilde{G})$ onto $M_{d(\delta)}(\mathbb{C})$ defines a spherical Fourier transform of type δ on $\tilde{G}$.*

Proof.
Let $f \in I_{c,\delta}(G)$ and $\chi_\delta * f_K = f$.

$$\begin{aligned}
\varphi_\delta(f) &= \int_G \langle \hat{z}, \tilde{g}\rangle \, \theta_f^\delta(\tilde{g}) d\tilde{g} \\
&= \int_G \langle \hat{z}, \tilde{g}\rangle \left(\int_K u_{\check{\delta}}(k^{-1}) f(k, k\cdot\tilde{g}) dk \right) d\tilde{g} \\
\varphi_\delta(f) &= \int_G \int_K \langle \hat{z}, \tilde{g}\rangle u_{\check{\delta}}(k^{-1}) f(k, k\cdot\tilde{g}) dk d\tilde{g} \\
\varphi_\delta(f) &= \int_G \int_K \langle \hat{z}, \tilde{g}\rangle u_{\check{\delta}}(k^{-1}) \chi_\delta * f_K(k, k\cdot\tilde{g}) dk d\tilde{g} \\
&= \int_G \int_K \int_K \langle \hat{z}, \tilde{g}\rangle u_{\check{\delta}}(k^{-1}) \chi_\delta(\tilde{k}) f_K(\tilde{k}k, k\tilde{k}\cdot\tilde{g}) dk dk d\tilde{g} \\
&= \int_G \int_K \int_K \langle \hat{z}, \tilde{g}\rangle u_{\check{\delta}}(k^{-1}) \chi_\delta(\tilde{k}) \left(\int_K f(k_1^{-1}k_1\tilde{k}k, k_1k\tilde{k}\cdot\tilde{g}) dk_1 \right) d\tilde{k} dk d\tilde{g} \\
&= \int_G \int_K \int_K \int_K \langle \hat{z}, \tilde{g}\rangle u_{\check{\delta}}(k^{-1}) \chi_\delta(\tilde{k})\, f(\tilde{k}k, k_1\tilde{k}k\cdot\tilde{g}) dk_1 d\tilde{k} dk d\tilde{g} \\
&= \int_G \int_K \int_K \int_K \langle \hat{z}, \tilde{g}\rangle u_{\check{\delta}}(k^{-1}) \chi_\delta(k_1^{-1}\tilde{k}_3k_1k^{-1})\, f(k_1^{-1}\tilde{k}_3k_1, \tilde{k}_3k_1\cdot\tilde{g}) dk_1 d\tilde{k}_3 dk d\tilde{g} \\
&= \int_G \int_K \int_K \langle \hat{z}, \tilde{g}\rangle\, f(\tilde{k}, \tilde{k}k_1\cdot\tilde{g}) \left(\int_K u_{\check{\delta}}(k^{-1}) \chi_\delta(k^{-1}k_1^{-1}\tilde{k}k_1) dk \right) dk_1 dk d\tilde{g} \\
&= \int_G \int_K \int_K \langle \hat{z}, \tilde{g}\rangle\, f(\tilde{k}, \tilde{k}k_1\cdot\tilde{g}) u_{\check{\delta}}(k_1^{-1}\tilde{k}k_1) dk_1 d\tilde{k} d\tilde{g} \\
&= \int_G \int_K f(\tilde{k}, \tilde{g}) \left(\int_K \langle \hat{z}, k^{-1}\cdot\tilde{g}\rangle u_{\check{\delta}}(k_1^{-1}\tilde{k}k_1) dk_1 \right) d\tilde{k} d\tilde{g}
\end{aligned}$$

By setting $\Phi_\delta(\tilde{k}, \tilde{g}) = \int_K \langle \hat{z}, k^{-1}\cdot\tilde{g}^{-1}\rangle u_{\check{\delta}}(k_1^{-1}\tilde{k}^{-1}k_1) dk_1$, we have

$$\varphi_\delta(f) = \int_G \int_K f(\tilde{k}, \tilde{g})\, \Phi_\delta\left((\tilde{k}, \tilde{g})^{-1}\right) d\tilde{k} d\tilde{g}.$$

We prove now that the function Φ_δ defined on $\tilde{G}$ is a spherical function of type δ. We first show that Φ_δ is K-central.

$$\begin{aligned}
\Phi_\delta(k(\tilde{k}, \tilde{g})k^{-1}) &= \Phi_\delta(k\tilde{k}k^{-1}, k\cdot\tilde{g}) \\
&= \int_K \langle \hat{z}, k_1^{-1}\cdot(k\cdot\tilde{g}^{-1})\rangle u_{\check{\delta}}(k_1^{-1}(k\tilde{k}k^{-1})^{-1}k_1) dk_1 \\
&= \int_K \langle \hat{z}, k_1^{-1}k\cdot\tilde{g}^{-1}\rangle u_{\check{\delta}}(k_1k\tilde{k}^{-1}k^{-1}k_1) dk_1
\end{aligned}$$

$$
\begin{aligned}
&= \int_K \langle \hat{z}, k_1^{-1}\cdot \tilde{g}^{-1}\rangle u_{\check{\delta}}(k_1^{-1}\tilde{k}^{-1}k_1)dk_1 \\
&= \Phi_\delta(\tilde{k},\tilde{g}).
\end{aligned}
$$

Therefore, Φ_δ is K-central.
We have

$$
\begin{aligned}
\chi_\delta * \Phi_\delta(\tilde{k},\tilde{g}) &= \int_K \chi_\delta(k)\,\Phi_\delta((\tilde{k},\tilde{g})k^{-1})dk \\
&= \int_K \chi_\delta(k)\,\Phi_\delta(k^{-1}\tilde{k},\tilde{g})dk \\
&= \int_K \chi_\delta(k)\left(\int_K \langle \hat{z}, k^{-1}\cdot \tilde{g}^{-1}\rangle u_{\check{\delta}}(k_1^{-1}(k^{-1}\tilde{k})^{-1}k_1)dk_1\right)dk \\
&= \int_K \chi_\delta(k)\left(\int_K \langle \hat{z}, k_1^{-1}\cdot \tilde{g}^{-1}\rangle u_{\check{\delta}}(k_1^{-1}\tilde{k}^{-1}kk_1)dk_1\right)dk \\
&= \int_K\int_K \chi_\delta(k)\langle \hat{z}, k_1^{-1}\cdot \tilde{g}^{-1}\rangle u_{\check{\delta}}(k_1^{-1}\tilde{k}^{-1}kk_1)dk_1dk \\
&= \int_K\int_K \chi_\delta(k\vec{k}_3^{-1})\langle \hat{z}, k_1^{-1}\cdot \tilde{g}^{-1}\rangle u_{\check{\delta}}(k_1^{-1}\tilde{k}_3^{-1}k_1)dk_1d\tilde{k}_3 \\
&= \int_K\int_K \chi_\delta\left((\tilde{k}_3k^{-1})^{-1}\right))\,\langle \hat{z}, k_1^{-1}\cdot \tilde{g}^{-1}\rangle u_{\check{\delta}}(k_1^{-1}\tilde{k}_3^{-1}k_1)dk_1d\tilde{k}_3 \\
&= \int_K\int_K \chi_{\check{\delta}}(\tilde{k}_3\vec{k}^{-1})\,\langle \hat{z}, k_1^{-1}\cdot \tilde{g}^{-1}\rangle u_{\check{\delta}}(k_1^{-1}\tilde{k}_3^{-1}k_1)dk_1d\tilde{k}_3 \\
&= \int_K \langle \hat{z}, k_1^{-1}\cdot \tilde{g}^{-1}\rangle d(\tilde{\delta})u_{\check{\delta}}(k_1^{-1})\left(\int_K u_{\check{\delta}}(\tilde{k}_3^{-1})\,tr(u_{\check{\delta}}(\tilde{k}_3)u_{\check{\delta}}(\tilde{k}^{-1}))\,d\tilde{k}_3\right)u_{\check{\delta}}(k_1)dk_1 \\
&= \int_K \langle \hat{z}, k_1^{-1}\cdot \tilde{g}^{-1}\rangle u_{\check{\delta}}(k_1^{-1})u_{\check{\delta}}(\tilde{k}^{-1})u_{\check{\delta}}(k_1)dk_1 \\
&= \int_K \langle \hat{z}, k_1^{-1}\cdot \tilde{g}^{-1}\rangle u_{\check{\delta}}(k_1^{-1}\tilde{k}^{-1}k_1)dk_1 \\
&= \Phi_\delta(\tilde{k},\tilde{g}). \qquad \text{For all } \tilde{k}\in K \text{ and } \tilde{g}\in G.
\end{aligned}
$$

So $\chi_\delta * \Phi_\delta(\tilde{k},\tilde{g}) = \Phi_\delta(\tilde{k},\tilde{g})$.
Furthermore Φ_δ is quasi-bounded. In fact,

$$
\begin{aligned}
\left\|\Phi_\delta(\tilde{k},\tilde{g})\right\| &= \left\|\int_K \langle \hat{z}, k_1^{-1}\cdot \tilde{g}^{-1}\rangle u_{\check{\delta}}(k_1^{-1}\tilde{k}^{-1}k_1)dk_1\right\| \\
&\le \int_K \left|\langle \hat{z}, k_1^{-1}\cdot \tilde{g}^{-1}\rangle\right|\,\left\|u_{\check{\delta}}(k_1^{-1}\tilde{k}^{-1}k_1)\right\|dk_1 \\
\left\|\Phi_\delta(\tilde{k},\tilde{g})\right\| &\le M\,\rho(\tilde{g}). \qquad \text{For all } \tilde{k}\in K \text{ and } \tilde{g}\in G.
\end{aligned}
$$

Finally we prove that the map $f \mapsto \varphi_\delta(f)$ is an irreducible representation of $I_{c,\delta}(\tilde{G})$.
Let f and $g \in I_{c,\delta}(\tilde{G})$.

$$
\begin{aligned}
\varphi_\delta(f*g) &= \int_G \langle \hat{z}, x\rangle\,\theta^\delta_{f*g}(x)d_G(x) \\
&= \int_G \langle \hat{z}, x\rangle\,\theta^\delta_{f*g}(x)d_G(x)
\end{aligned}
$$

$$
\begin{aligned}
&= \int_G \int_G \langle \hat{z}, x\rangle\, \theta_f^\delta\left(x\, y^{-1}\right)\, \theta_g^\delta(y) d_G(x) d_G(y) \\
&= \int_G \left(\int_G \langle \hat{z}, x\rangle\, \theta_f^\delta(x) d_G(x) \right) \langle \hat{z}, y\rangle\, \theta_g^\delta(y) d_G(y) \\
&= \int_G \varphi_\delta(f) \langle \hat{z}, y\rangle\rangle\, \theta_g^\delta(y) d_G(y) \\
&= \varphi_\delta(f)\, \varphi_\delta(g).
\end{aligned}
$$

Therefore the map $f \mapsto \varphi_\delta(f)$ is a representation of the algebra $I_{c,\delta}(\tilde{G})$. Since $u_{\check{\delta}}$ is irreducible, then $f \mapsto \varphi_\delta(f)$ is irreducible. So $f \mapsto \varphi_\delta(f)$ is a spherical Fourier transform of type δ. □

Remark 2.6. If δ is the trivial one dimensional and $\hat{z}$ a unitary character on G, then

$$
\varphi(f) = \int_G \int_K \langle \hat{z}, g\rangle\, f(k, k \cdot g)\, dg\, dk = \int_G \langle \hat{z}, g\rangle\, \theta_f(g)\, dg
$$

where the map $f \longmapsto \theta_f$ defined from $I_c(K \rtimes G)$ into $I_c(G)$ is an Abel transformation.

Example

Let $H_2 = \mathbb{C}^2 \times \mathbb{R}$ a Heisenberg group of dimension 5. The group law is defined by:

$$
(z, t)(z', t') = \Big(z + z', t + t' + \frac{1}{2} Im(z\, \bar{z}')\Big)
$$

where Im designate the imaginary part. H_2 is a connected Lie group. Let $\omega \in \mathbb{C}^2$, we define a character χ_ω in H_2 by:

$$
\chi_\omega(z, t) = e^{i\, Re\langle \omega, z\rangle}
$$

where Re designate the real part.

Let $K = SU(2)$ be the special unitary group of order 2.
An element of $SU(2)$ is written

$$
U_{a,b} = \begin{pmatrix} 0 & -\bar{b} \\ b & \bar{a} \end{pmatrix}
$$

where $a, b \in \mathbb{C}$ such that $|a|^2 + |b|^2 = 1$.
We observe that $U_{a,b}^{-1} = U_{a,b}^* = U_{\bar{a},-b}$ and the action of $U_{a,b}$ in $\mathbb{C}^2$ is given by:

$$
U_{a,b} \cdot z = U_{a,b} \cdot \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} a\, z_1 - \bar{b}\, z_2 \\ b\, z_1 + \bar{a}\, z_2 \end{pmatrix}.
$$

The action of $SU(2)$ on H_2 is:

$$
U_{a,b} \cdot (z, t) = \Big(U_{a,b} \cdot z, t\Big).
$$

Let $\mathcal{P}$ be the polynomial space on $\mathbb{C}^2$ and W_n be a subspace of $\mathcal{P}$ of dimension n. We have:

$$
W_n = \{P \in \mathcal{P} : P(z_1, z_2 = \sum_{i=0}^{n} C_i\, z_1^i\, z_2^{n-i}; C_i \in \mathbb{C}\}.
$$

The space W_n is of dimension $n + 1$.
Let $\tilde{\pi}$ a representation of $SU(2)$ defined by:

$$
\begin{aligned}
\Big(\tilde{\pi}(U_{a,b}) P\Big)\Big(z_1, z_2\Big) &= P\Big(U_{a,b}^{-1} \cdot (z_1, z_2)\Big) \\
&= P\Big(a z_1 - \bar{b} z_2, b z_1 + \bar{a} z_2\Big).
\end{aligned}
$$

By putting $\tilde{\pi}_{|W_n} = \tilde{\pi}_n$, $\tilde{\pi}_n$ is irreducible for all $n>0$.
Let $\tilde{G} = SU(2) \rtimes H_2$ and $K = SU(2)$. Let χ_ω be a character on H_2 and $\tilde{\pi}_n$ a representation on $SU(2)$.
Then, the function $\Phi^{\langle n \rangle}$ defined on $\tilde{G} = SU(2) \rtimes H_2$ by:

$$\Phi^{\langle n \rangle}\Big(U_{a,b}, (z,t)\Big) = \int_{SU(2)} \chi_\omega\Big(U_{a_1,b_1}, (z,t)\Big)\, \tilde{\pi}_n\Big(U^{-1}_{a_1,b_1}\, U_{a,b}\, U_{a_1,b_1}\Big) da_1\, db_1$$

is a spherical function of type $\tilde{\pi}_n$.
Let $f \in I_{c,\delta}\Big(SU(2) \rtimes H_2, End(W_n)\Big)$. Then, the Abel transform of f of type $\tilde{\pi}_n$ is defined by:

$$\theta^{\langle n \rangle}_f(z,t) = \int_{SU(2)} \tilde{\pi}_n\Big(U^{-1}_{a,b}\Big)\, f\Big(U_{a,b},\, U_{a,b}\cdot(z,t)\Big)\, da\, db$$
$$\theta^{\langle n \rangle}_f(z,t) = \int_{SU(2)} \tilde{\pi}_n\Big(U_{a,b}\Big)\, f\Big(U^{-1}_{a,b},\, U^{-1}_{a,b}\cdot(z,t)\Big)\, da\, db.$$

Thanks to the Abel transformation of type $\tilde{\pi_n}$ defined on $\tilde{G}$, the spherical Fourier transform of type π_n is then defined by:

$$\begin{aligned}\mathcal{F}(f)(\Phi^{\langle n \rangle}) &= \int_{H_2} \chi_\omega(z,t)\, \theta^{\langle n \rangle}_f(z,t)\, dz\, dt \\ &= \int_{H_2} \chi_\omega(z,t)\Bigg(\int_{SU(2)} \tilde{\pi}_n\Big(U^{-1}_{a,b}\Big)\, f\Big(U_{a,b}, U_{a,b}\cdot(z,t)\Big)\, da\, db\Bigg) dz\, dt \\ &= \int_{SU(2)\rtimes H_2} \chi_\omega(z,t)\, \tilde{\pi}_n\Big(U^{-1}_{a,b}\Big)\, f\Big(U_{a,b}, U_{a,b}\cdot(z,t)\Big)\, da\, db\, dz\, dt.\end{aligned}$$

2.1. Generalized Abel Transformation on reductive Lie group

Let G be a locally compact group and K be a compact subgroup of G. Let us consider $I(G)$, the space of continuous complex-valued functions on G with compact support. By identifying χ_δ with a bounded measure on G, we set for any function $f \in I_c(G)$, ${}_\delta f(x) = \bar{\chi_\delta} * f(x) = \int_K \chi_\delta(k)\, f(kx) dk$ and $f_\delta(x) = f * \chi_\delta(x) = \int_K \chi_\delta(k^{-1})\, f(xk) dk$. Let $I_\delta(G)$ be the subspace of $I(G)$ consisting of all functions δ-invariants i.e ($I_\delta(G) = \{f \in I(G), f = {}_\delta f = f_\delta\}$) and $J_c(G) = \{f \in I(G), f(xk) = f(kx), k \in K, x \in G\}$ the set of functions $f \in I(G)$ which are K-central. $J_c(G)$ is a subalgebra of $I(G)$. The map $f \mapsto f_K$ defined by:

$$f_K(x) = \int_K f(k\,x\,k^{-1}) dk$$

is a projection of $I(G)$ onto $J_c(G)$.
We set: $I_{c,\delta}(G) = J_c(G) \cap I_\delta(G)$. $I_{c,\delta}(G)$ is a subalgebra of $I(G)$ and the map $f \mapsto \bar{\chi_\delta} * f_K$ is a projection of $I(G)$ onto $I_{c,\delta}(G)$. If δ is a trivial representation class of dimension 1 of K, then $I_{c,\delta}(G)$ is thus identified with the algebra $I(G)$ of continuous complex valued functions on G with compact support and biinvariant by K.
Let G be a reductive Lie group with Lie algebra $\mathcal{G}$. It is well-known that $\mathcal{G} = \mathscr{Z} \oplus [\mathcal{G}, \mathcal{G}] = \mathscr{Z} \oplus \bar{\mathcal{G}}$ where $\mathscr{Z}$ is the center of the Lie algebra $\mathcal{G}$ and $\bar{\mathcal{G}}$ is semisimple.

Let Z and G be the analytic subgroups associated to $\mathscr{Z}$ and $\bar{\mathcal{G}}$. Let $\bar{G}$ be a connected semisimple Lie group with finite center and, $\bar{G} = KAN^+$, an Iwasawa decomposition of $\bar{G}$, where K a maximal compact subgroup, A is an abelian subgroup and N^+ is a nilpotent subgroup. Let $\mathcal{A} = \mathrm{Lie}(A)$. For all $\bar{x} \in \bar{G}$, $\bar{x} = \kappa(\bar{x}) \exp(H(\bar{x}))\, n(\bar{x})$, where $\kappa(\bar{x}) \in K$, $H(\bar{x}) \in \mathcal{A}$ and $n(\bar{x}) \in N^+$.

G is written as semi-direct product of Z and $\bar{G}$, $G = Z \rtimes \bar{G}$.

Any x belonging to G is of the form $x=(z,\bar{x})$, where $z\in Z$, $\bar{x}\in\bar{G}$. The action of K on G is defined by:

$$\begin{aligned}&\alpha: K\times G\longrightarrow G\\&(k,(z,\bar{x}))\longmapsto k\cdot(z,\bar{x})=(z,k\cdot\bar{x}) \text{ with } k\cdot\bar{x}=k\,\bar{x}\,k^{-1}.\end{aligned}$$

Let $X(Z)$ be the set of characters of Z i.e the set of all irreducible representations of Z of dimension 1. Let U be a representation of G. Then, there exists a representation T of Z and a representation L of $\bar{G}$ such that $U(z,\bar{x})=T(z)\,L(\bar{x})$. If U is irreducible then, $U(z,\bar{x})=(\hat{z},z)L(\bar{x})$, where $\hat{z}$ is a character of Z (See [1]). The normalized Haar measure on $\bar{G}$ is defined by $d_{\bar{G}}(\bar{x})=h^{2\rho}\,dk\,d_A(h)\,d_{N^+}(n).(x\in\bar{G},\bar{x}=k\,h\,n)$ where ρ denotes the sum of positives half-roots associated to the pair $(\bar{G},\mathcal{A})$. We consider an invariant measure dx on G such that $dx=dz\,d\bar{x}$, where $x=(z,\bar{x})$. Let $\nu\in\mathcal{A}^*$ and let Ψ_ν be a zonal spherical function on $\bar{G}$ then:

$$\int_{\bar{G}} f(\bar{x})\,\Psi_\nu(\bar{x})\,dx=\int_A \bar{F}_f(h)\,h^{i\nu}\,d_A(h),\ f\in I_c(\bar{G})$$

where the function $f\mapsto\bar{F}_f$ from $I_c(\bar{G})$ into $I_c(A,\mathbb{C})$ is the Abel transformation on $\bar{G}$ (See [14]). A spherical function on G is of the form $\varphi(z,\bar{x})=(\hat{z},z)\,\Psi_\nu(\bar{x})$, where Ψ_ν is a zonal spherical function on G.

Definition 2.7. Let $f\in I_c(G)$.
The Abel transformation: $f\longmapsto F_f$ is a linear mapping from $I_c(G)$ into $I_c(Z\times A,\mathbb{C})$ where F_f defined by:

$$\text{for all } (z,h)\in Z\times A,\ F_f(z,h)=h^\rho\langle\hat{z},z^{-1}\rangle\int_{N^+} f(z,h\,n)\,d_{N^+}(n).$$

Theorem 2.8. Let $f\in I_c(G)$ and $\varphi=\langle\hat{z},\cdot\rangle\,\Psi_\nu(\cdot)$ be a spherical function on G. Then,

$$\int_G f(x)\,\varphi(x^{-1})\,dx=\int_Z\int_A h^{i\nu}F_f(z,h)\,d_A(h)\,dz.$$

Proof. Let $\mu\in\mathcal{A}^*$ such that $\mu=-\rho-i\nu$, $\Psi_\nu(\bar{x})=\int_K e^{-(\mu+2\rho)(H(\bar{x}^{-1}k))}\,dk$ with $\hat{z}\in X(Z)$. Let $f\in I_c(G)$. It follows that

$$\begin{aligned}\int_G f(x)\,\varphi(x^{-1})\,dx&=\int_Z\int_G f(z,\bar{x})\,\varphi(z^{-1},\bar{x}^{-1})\,d\bar{x}\,dz\\&=\int_Z\int_G\int_K f(z,\bar{x})\langle\hat{z},z^{-1}\rangle\,e^{-(\mu+2\rho)(H(\bar{x}\,k))}\,dk\,d\bar{x}\,dz\\&=\int_Z\int_K\int_G f(z,\bar{x}k)\langle\hat{z},z^{-1}\rangle\,e^{-(\mu+2\rho)(H(\bar{x}\,k))}\,dz\,dk\,d\bar{x}\\&=\int_Z\int_G\int_K f(z,\bar{x})\langle\hat{z},z^{-1}\rangle\,e^{-(\mu+2\rho)(H(\bar{x}))}\,dk\,d\bar{x}\,dz\\&=\int_Z\int_G f(z,\bar{x})\langle\hat{z},z^{-1}\rangle\,e^{-(\mu+2\rho)(H(\bar{x}))}\,d\bar{x}\,dz\\&=\int_Z\int_K\int_A\int_{N^+} f(z,k\,h\,n)\langle\hat{z},z^{-1}\rangle\,e^{-\mu-2\rho}\,h^{2\rho}\,dz\,d_A(h)\,d_{N^+}\,dk\\&=\int_Z\int_K\int_A\int_{N^+} f(z,k,h\,n)\langle\hat{z},z^{-1}\rangle\,h^{-\mu}\,dz\,d_A(h)\,d_{N^+}\,dk\end{aligned}$$

$$= \int_Z \int_K \int_A \int_{N^+} f(z, h\,n)\langle \hat{z}, z^{-1}\rangle\, h^{i\nu}\, h^{\rho}\, dz\, dk\, d_A(h)\, d_{N^+}$$
$$= \int_Z \int_A \int_{N^+} f(z, h\,n)\langle \hat{z}, z^{-1}\rangle\, h^{i\nu}\, h^{\rho}\, dz\, d_A(h)\, d_{N^+}$$
$$= \int_Z \int_A h^{i\nu} \left(h^{\rho})\langle \hat{z}, z^{-1}\rangle \int_{N^+} f(z, h\,n)\, d_{N^+} \right) d_A(h)\, dz$$

The mapping $f \mapsto F_f$ from $I_c(G)$ into $I_c(Z \times A, \mathbb{C})$ is the extension of the Abel transformation on reductive Lie group G and F_f is the Abel transform of f. □

Let $U_{c,\delta}(G)$ be the space of μ-spherical function on G where $\mu = (u_{\check{\delta}}, u_{\check{\delta}})$ is a double Banach representation of K. Thanks to identification of $I_{c,\delta}(G)$ and $U_{c,\delta}(G)$ by the isomorphism $f \mapsto \Psi_f^{\delta}$ with
$\Psi_f^{\delta}(z, \bar{y}) = \int_K f(z, k\,\bar{y})\, u_{\check{\delta}}(k^{-1})dk$, we have a generalization of the Abel transformation by putting:

$$F_f^{\delta}(z, h) = h^{\rho}\langle \hat{z}, z^{-1}\rangle \int_{N^+} \Psi_f^{\delta}(z, h\,n)\, d_{N^+}(n)$$
$$= h^{\rho}\langle \hat{z}, z^{-1}\rangle \int_{N^+} \left(\int_K f(z, k\,h\,n)\, u_{\check{\delta}}(k^{-1})dk \right) d_{N^+}(n)$$
$$F_f^{\delta}(z, h) = h^{\rho}\langle \hat{z}, z^{-1}\rangle \int_{N^+} \int_K f(z, k\,h\,n)\, u_{\check{\delta}}(k^{-1})dk\, d_{N^+}(n)$$

Definition 2.9. Let $\delta \in \hat{K}$, then the function F_f^{δ} is called the generalized Abel transform (of type δ) of f.

Theorem 2.10. Let $\mu \in \mathcal{A}^*$, the map φ_δ defined on $I_{c,\delta}(G)$ by

$$\varphi_\delta(f) = \int_Z \int_A h^{\mu+\rho}\, F_f^{\delta}(z, h)\, d_A(h)\, dz$$

defines a spherical Fourier transform of type δ on G.

Proof.

$$\varphi_\delta(f) = \int_Z \int_A h^{\mu+\rho}\, F_f^{\delta}(z, h)\, d_A(h)\, dz$$
$$= \int_Z \int_A h^{\mu+\rho} \left(h^{\rho}\langle \hat{z}, z^{-1}\rangle \int_{N^+} \int_K f(k\,h\,n)\, u_{\check{\delta}}(k^{-1})dk\, d_{N^+}(n) \right) d_A(h)\, dz$$
$$= \int_Z \int_A \int_{N^+} \int_K \langle \hat{z}, z^{-1}\rangle h^{\mu+2\rho}\, f(z, k\,h\,n)\, u_{\check{\delta}}(k^{-1})dk\, d_{N^+}(n) d_A(h)\, dz$$
$$= \int_Z \int_G \langle \hat{z}, z^{-1}\rangle\, h^{\mu}\, f(z, \bar{y})\, u_{\check{\delta}}(\kappa(\bar{y}^{-1}))\, d\bar{y}\, dz$$
$$= \int_Z \int_G \int_K \langle \hat{z}, z^{-1}\rangle\, h^{\mu}\, f(z, k\,\bar{y}\,k^{-1})\, u_{\check{\delta}}(\kappa(\bar{y}^{-1}))\, d\bar{y}\, dz\, dk$$
$$= \int_Z \int_{\bar{G}} \int_K \langle \hat{z}, z^{-1}\rangle\, h^{\mu}\, f(z, \bar{y})\, u_{\check{\delta}}(\kappa(k^{-1}\bar{y}^{-1}\,k))\, dk\, d\bar{y}\, dz$$
$$= \int_Z \int_{\bar{G}} f(z, \bar{y}) \left(\langle \hat{z}, z^{-1}\rangle\, h^{\mu} \int_K u_{\check{\delta}}(\kappa(k^{-1}\bar{y}^{-1}\,k))dk \right) d\bar{y}\, dz.$$

Let us put

$$\Phi_{\delta,\mu}(z, \bar{y}) = \langle \hat{z}, z^{-1}\rangle\, h^{\mu} \int_K u_{\check{\delta}}(\kappa(k^{-1}\,\bar{y}^{-1}\,k))dk$$

$$= \langle \hat{z}, z^{-1} \rangle \int_K u_{\check{\delta}}(\kappa(k^{-1}\,\bar{y}^{-1}\,k))\, e^{\mu(H(\bar{y}^{-1}\,k))} dk$$

$$\varphi_\delta(f) = \int_Z \int_G f(z,\bar{y})\, \Phi_{\delta,\mu}(z,\bar{y}) dz\, d\bar{y}.$$ It follows that $\varphi_\delta(f) = \int_Z \int_G f(z,\bar{y})\, \Phi_{\delta,\mu}(z,\bar{y}) dz\, d\bar{y}.$

We prove now that the function $\Phi_{\delta,\mu}$ defined on G is a spherical function of type δ.
We first show that $\Phi_{\delta,\mu}$ is K-central.

$$\begin{aligned}
(\Phi_{\delta,\mu}(z,\bar{y}))_K &= \int_K \Phi_{\delta,\mu}(z, \bar{k}\,\bar{y}\,\bar{k}^{-1}) d\bar{k} \\
&= \int_K \langle \hat{z}, z^{-1} \rangle \left(\int_K u_{\check{\delta}}(\kappa(k^{-1}\,\bar{k}\,\bar{y}\,\bar{k}^{-1}\,k))\, e^{\mu(H(\bar{k}\,\bar{y}^{-1}\,\bar{k}^{-1}\,k))}\, dk \right) d\bar{k} \\
&= \langle \hat{z}, z^{-1} \rangle \int_K \int_K u_{\check{\delta}}(\kappa(k^{-1}\,\bar{k}\,\bar{y}\,\bar{k}^{-1}\,k))\, e^{\mu(H(\bar{k}\bar{y}^{-1}\,\bar{k}^{-1}\,k))} d\bar{k}\, dk \\
&= \langle \hat{z}, z^{-1} \rangle \int_K \int_K u_{\check{\delta}}(\kappa(t\,\bar{y}\,t^{-1}))\, e^{\mu(H(\bar{y}^{-1}\,t^{-1}))} dt\, d\bar{k} \\
&= \langle \hat{z}, z^{-1} \rangle \int_K \int_K u_{\check{\delta}}(\kappa(t^{-1}\,\bar{y}\,t))\, e^{\mu(H(\bar{y}^{-1}\,t))} dt\, d\bar{k} \\
&= \int_K \left(\langle \hat{z}, z^{-1} \rangle \int_K u_{\check{\delta}}(\kappa(t^{-1}\,\bar{y}\,t))\, e^{\mu(H(\bar{y}^{-1}\,t))} dt \right) d\bar{k} \\
&= \int_K \Phi_{\delta,\mu}(z,\bar{y}) d\bar{k} \\
&= \Phi_{\delta,\mu}(z,\bar{y})
\end{aligned}$$

Therefore, $\Phi_{\delta,\mu}$ is K-central.
We have

$$\begin{aligned}
\chi_\delta * \Phi_{\delta,\mu}(z,\bar{y}) &= \langle \hat{z}, z^{-1} \rangle \int_K \chi_\delta(\tilde{k}^{-1}) \int_K u_{\check{\delta}}(\kappa(k^{-1}\,\tilde{k}\,\bar{y}\,\tilde{k}^{-1}\,k))\, e^{\mu(H(\tilde{k}\,\bar{y}^{-1}\,\tilde{k}\,k))} dk\, d\tilde{k} \\
&= \langle \hat{z}, z^{-1} \rangle \int_K d(\delta) \int_K u_{\check{\delta}}(k^{-1})\, u_{\check{\delta}}(\tilde{k})\, tr(u_{\check{\delta}}(\kappa(\tilde{k}^{-1}))\, U_{\check{\delta}}(\kappa(\bar{y}\,k)) e^{\mu(H(\bar{y}^{-1}\,k))}\, dk\, d\tilde{k} \\
&= \langle \hat{z}, z^{-1} \rangle \int_K u_{\check{\delta}}(k^{-1}) \left(d(\delta) \int_K u_{\check{\delta}}(\tilde{k})\, tr(u_\delta(k^{-1}))\, u_{\check{\delta}}(\kappa(\bar{y}\,k))\, e^{\mu(H(\bar{y}^{-1}\,k))} d\tilde{k} \right) dk \\
&= \int_K \langle \hat{z}, z^{-1} \rangle u_{\check{\delta}}(k^{-1})\, u_{\check{\delta}}(\kappa(\bar{y}\,k))\, e^{\mu(H(\bar{y}^{-1}\,k))}\, dk \\
&= \langle \hat{z}, z^{-1} \rangle \int_K u_{\check{\delta}}(\kappa(k^{-1}\,\bar{y}\,k))\, e^{\mu(H(\bar{y}^{-1}\,k))}\, dk \\
&= \Phi_{\delta,\mu}(z,\bar{y})
\end{aligned}$$

So, $\chi_\delta * (\Phi_{\delta,\mu}(z,\bar{y})) = \Phi_{\delta,\mu}(z,\bar{y})$

We prove now that $\Phi_{\delta,\mu}$ is quasi-bounded. Denote by $B_\delta = \{e_1, e_2, \cdots e_{d(\delta)}\}$ a basis of E_δ. A matrix of $U_{\check{\delta}}(k)$ in the basis B_δ is: $(b_{ij}(k))_{1 \le i,j \le d(\delta)}$.

$$b_{ij}(k) = \langle u_{\check{\delta}}(k) e_i, e_j \rangle$$

$$= \langle e_i, u_{\check{\delta}}(k^{-1}) e_j \rangle$$

$$b_{ij}(k) = \overline{b_{ji}(k^{-1})}$$

For all $1 \leq i, j \leq d(\delta)$

$$
\begin{aligned}
(\Phi_{\delta,\mu}(z,\bar{y}))_{ij} &= \langle \hat{z}, z^{-1}\rangle \int_K \left(u_\delta(\kappa(k^{-1}\,\bar{y}\,k))\right)_{ij} e^{\mu(H(\bar{y}\,k))}\,dk \\
&= \langle \hat{z}, z^{-1}\rangle \int_K \sum_{t=1}^{d(\delta)} b_{it}(k^{-1})\,b_{tj}(\kappa(\bar{y}\,k))\,e^{\mu(H(\bar{y}\,k))}\,dk \\
&= \sum_{t=1}^{d(\delta)} \langle \hat{z}, z^{-1}\rangle \int_K b_{it}(k^{-1})\,b_{tj}(\kappa(\bar{y}\,k))\,e^{\mu(H(\bar{y}\,k))}\,dk \\
&= \sum_{t=1}^{d(\delta)} \langle \hat{z}, z^{-1}\rangle \int_K \overline{b_{ti}(k)}\,b_{tj}(\kappa(\bar{y}\,k))\rangle\,e^{\mu(H(\bar{y}\,k))}\,dk \\
&= \langle \hat{z}, z^{-1}\rangle \sum_{t=1}^{d(\delta)} \left(U^{-\bar{\mu}-2\rho}(\bar{y}^{-1})b_{tj},\, b_{ti}\right) \\
&= \langle \hat{z}, z^{-1}\rangle \sum_{t=1}^{d(\delta)} (b_{tj}, U^{\mu}(\bar{y})\,b_{ti}) \\
(\Phi_{\delta,\mu}(z,\bar{y}))_{ij} &= \langle \hat{z}, z^{-1}\rangle \sum_{t=1}^{d(\delta)} (b_{tj}, U^{\mu}(\bar{y})\,b_{ti}) \\
|(\Phi_{\delta,\mu}(z,\bar{y}))_{ij}| &= \left| \langle \hat{z}, z^{-1}\rangle \sum_{t=1}^{d(\delta)} (b_{tj}, U^{\mu}(\bar{y})\,b_{ti}) \right| \\
&\leq \left|\langle \hat{z}, z^{-1}\rangle\right| \left| \sum_{i=1}^{d(\delta)} (b_{tj}, U^{\mu}(\bar{y})\,b_{ti}) \right| \\
&\leq \left|\langle \hat{z}, z^{-1}\rangle\right| \sum_{t=1}^{d(\delta)} \|b_{tj}\|_{L^2(K)}\ \|U^{\mu}(\bar{y})\,b_{ti}\|_{L^2(K)} \\
&\leq \left|\langle \hat{z}, z^{-1}\rangle\right| \times \|U^{\mu}(\bar{y})\| \sum_{t=1}^{d(\delta)} \|b_{tj}\|_{L^2(K)}\,\|b_{ti}\|_{L^2(K)} \\
\|\,(\Phi_{\delta,\mu}(z,\bar{y}))\,\| &\leq \left|\langle \hat{z}, z^{-1}\rangle\right| \times \|U^{\mu}(\bar{y})\| \left(\sum_{i,j}^{d(\delta)} \sum_{t=1}^{d(\delta)} \|b_{tj}\|_{L^2(K)}\,\|b_{ti}\|_{L^2(K)} \right) \\
&\implies \frac{\|\Phi_{\delta,\mu}(z,\bar{y})\|}{\|U^{\mu}(\bar{y})\|} < +\infty
\end{aligned}
$$

The quasi-boundedness of $\Phi_{\delta,\mu}$ is there by established.
Let us show that the map

$$f \mapsto \int f(z,\bar{y})\Phi_{\delta,\mu} dz\, d\bar{y}$$

($f \in I_{c,\delta}(G)$) is a representation of the algebra $I_{c,\delta}(G)$

$$
\begin{aligned}
\varphi_\delta(f * g) &= \int_Z \int_A h^{\mu+\rho}\, F^{\delta}_{f*g}(z,h)\, d_A(h)\, dz \\
&= \int_Z \int_A h^{\mu+\rho}\, F^{\delta}_{f*g}(y)\, d_A(h)\, dz \\
&= \int_Z \int_A h^{\mu+\rho} \left(F^{\delta}_{f} * F^{\delta}_{g}\right)(y)\, d_A(h)\, d(z)
\end{aligned}
$$

$$
\begin{aligned}
&= \int_Z \int_A h^{\mu+\rho} \left(\int_Z \int_A F_f^\delta(yx^{-1})\, F_g^\delta(x)\, dx \right) d_A(h)\, dz \\
&= \int_Z \int_A \int_Z \int_A h^{\mu+\rho}\, F_f^\delta(z\bar{z}^{-1}, h\,\tilde{h}^{-1})\, F_g^\delta(\bar{z}, \tilde{h})\, d_A(\tilde{h})\, d\bar{z}\, d_A(h)\, dz \\
&= \int_Z \int_A \int_Z \int_A (h\,\tilde{h}^{-1})^{\mu+\rho}\, F_f^\delta(z\bar{z}^{-1}, h\,\tilde{h}^{-1})\, \tilde{h}^{\mu+\rho}\, F_g^\delta(\bar{z}, \tilde{h})\, d_A(\tilde{h})\, d\bar{z}\, d_A(h)\, dz \\
&= \int_Z \int_A \left(\int_Z \int_A (h\,\tilde{h}^{-1})^{\mu+\rho} \left(F_f^\delta(z\bar{z}^{-1}, h\,\tilde{h}^{-1}) \right) d_A(h)\, dz \right) h^{\mu+\rho}\, F_g^\delta(\bar{z}, \tilde{h})\, d_A(\tilde{h})\, d\bar{z} \\
&= \varphi_\delta(f)\, \varphi_\delta(g)
\end{aligned}
$$

which proves that φ_δ is a representation of $I_{c,\delta}(G)$. Therefore, $\Phi_{\delta,\mu}$ is a spherical function on G of type δ.
In conclusion, the map $f \mapsto \varphi_\delta(f)$ is a spherical Fourier transform of type δ of f. □

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