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A reproducing kernel associated with vector fields

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Abstract. Let G be a Lie group, $\mathfrak{X}(G)$ the space of infinitely differentiable vector fields on G, $C^\infty(G)$ the space of smooth complex-valued functions on G and let $\mathfrak{g}$ denote the Lie algebra of the group G. In this work, we consider a continuous evaluation functional on $\mathfrak{X}(G)$ together with an inner product on this space, with the aim of constructing a positive definite reproducing kernel defined on $C^\infty(G) \times C^\infty(G)$. This construction enables us to endow $\mathfrak{X}(G)$ with the structure of a reproducing kernel Hilbert space. Since $\mathfrak{g}$ can be viewed as a Lie subalgebra of $\mathfrak{X}(G)$, we also induce a structure of reproducing kernel Hilbert space on $\mathfrak{g}$. Subsequently, we define a linear transformation from $\mathfrak{g}$ into $\mathcal{F}(C^\infty(G))$, the space of complex-valued functions on $C^\infty(G)$ in order to establish both an inversion formula and the transform of a reproducing kernel associated with vector fields on G.

Key Words and Phrases: vector field, reproducing kernel, Lie algebra of Lie group.

2010 Mathematics Subject Classifications: 46E27, 46E22, 22E60.

1. Introduction

The concept of reproducing kernel was first introduced through the works of S. Bergman and S. Szegő (see [3, 17]). The theory was subsequently formalized and significantly developed by Nachman Aronszajn, whose contributions had a profound impact across various areas of mathematics. Reproducing kernel theory has since found applications in diverse fields such as machine learning, statistics, signal processing, quantum mechanics, and interpolation theory (see [6, 9, 13]). One of the foundational results established by Aronszajn is the bijective correspondence between an RKHS and its reproducing kernel K (see [2]). Specifically, if K is a positive definite kernel on an arbitrary set E, then there exists a unique RKHS H of functions on E such that $\{K(x,) : x \in E\} \subset H$. Numerous examples of RKHSs can be found in [1, 15, 16], and many researchers

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have investigated structural properties of these spaces (see [1, 19]), including tensor products, multipliers, and interpolations between different RKHSs. The study of linear transformations and inversion formulas within RKHSs was notably advanced by S. Saitoh (see [15, 16]). Beyond the classical theory, various generalizations of RKHS have been proposed to accommodate broader settings. For example, Canu et al. (2003) (see [5]) extended the concept to non-Hilbert spaces via kernel-based point evaluations. In 2008, Jaeseong Heo (see [8]) further extended the theory to C^*-modules, which generalize Hilbert spaces by allowing inner products to take values in a C^*-algebra. He studied reproducing kernels valued in a C^*-algebra. In 2009, Haizhang Zhang, Yuesheng Xu, and Jun Zhang (see [19]) generalized RKHS theory to Banach spaces by developing the framework of semi-inner-product reproducing kernel Banach spaces. This framework accommodates the absence of an inner product in Banach spaces and allows the study of fundamental properties of reproducing kernel Banach spaces. In our previous work [12, 11], we proposed an extension of RKHS theory to the Cartan subalgebra of a semisimple Lie algebra, using the Killing form to define a positive definite kernel. Since the Killing form is non-degenerate on semisimple Lie algebras, it naturally induces a valid RKHS structure.

In the present work, we propose a further extension of RKHS theory, focusing on the case of vector fields on a Lie group G. Specifically, we construct a reproducing kernel space on $\mathfrak{X}(G)$, the space of smooth vector fields on G, using a continuous evaluation functional. Following the approach of S. Saitoh (see [15, 16]), we establish an inversion formula for our kernel and characterize its transformation under a linear operator, interpreted here as a representation of the Lie algebra $\mathfrak{g}$. For this purpose, we draw on foundational results from Lie theory, vector fields, and representation theory, as found in [4, 10, 14]. This paper is organized as follows. Section 2 presents the preliminaries, and Section 3 contains the main results.

2. PRELIMINARIES

Definition 2.1 (See [7] p.11)**.** *Let (X,d) and $(Y,d^{'})$ be metric spaces and $f : X \to Y$ a map. f is said to be a Lipschitz map on X if there exists a constant $M \geq 0$ such that*

$$d^{'}(f(x),f(y)) \;\leq\; M\, d_X(x,y), \quad \forall x,y \in X.$$

We also say that f is an M-Lipschitz map.

The least such constant M is called the Lipschitz constant *of f, denoted*

$$\mathrm{Lip}(f) := \sup_{x\neq y} \frac{d^{'}(f(x),f(y))}{d(x,y)}.$$

f is called a contraction *mapping if* $\mathrm{Lip}(f) < 1$.

A *reproducing kernel Hilbert space* (RKHS) H on a set E is a Hilbert space of functions $f : E \to \mathbb{C}$ such that for every $x \in E$, the point evaluation functional

$$\varepsilon_x : H \to \mathbb{C}, \quad f \mapsto f(x)$$

is continuous.

By the Riesz-Fréchet theorem, there exists a unique function $K(\cdot, x) \in H$ for each $x \in E$ such that

$$f(x) = \langle f, K(\cdot, x)\rangle_H, \quad \forall f \in H.$$

The function $K : E \times E \to \mathbb{C}$ is called the *reproducing kernel* of H, and it satisfies

$$K(x, y) = \langle K(\cdot, y), K(\cdot, x)\rangle_H, \quad \forall x, y \in E.$$

Now let H be a Hilbert space with inner product $\langle \cdot, \cdot \rangle_H$. Let $\mathcal{F}(E)$ denote the set of complex-valued functions on E. Let $h : E \to H$ be a function defined by $h(p) = h_p$. Define a linear operator $L : H \to \mathcal{F}(E)$ by

$$(Lf)(p) = \langle f, h_p\rangle_H.$$

Define the kernel $K : E \times E \to \mathbb{C}$ by

$$K(p, q) = \langle h_q, h_p\rangle_H.$$

Let $R(L)$ denote the range of L. Define a norm on $R(L)$ by

$$\|\tilde{f}\|_{R(L)} = \inf\{\|f\|_H : Lf = \tilde{f}\}.$$

The above results concerning RKHS can be found in [1].

Theorem 2.2 (See [16], p.21)**.** *Let K be as defined above. Then $(R(L), \langle \cdot, \cdot \rangle_{R(L)})$ is a Hilbert space satisfying:*

1. *For each $q \in E$, the function $K(\cdot, q) \in R(L)$.*
2. *For every $f \in R(L)$ and every $q \in E$,*

$$\tilde{f}(q) = \langle \tilde{f}, K(\cdot, q)\rangle_{R(L)}.$$

Moreover, L is an isometry if and only if the family $\{h_p, p \in E\}$ is complete in H.

From this theorem, we see that the range of the linear transform is a reproducing kernel space.

Definition 2.3 (See [10]). *A group G is called a* Lie group *if:*

1. *G is a group;*

2. *G is an analytic manifold;*

2. *The map $(x, y) \mapsto xy^{-1}$ from $G \times G$ to G is analytic.*

Let G be a Lie group with identity element e, and T_eG the tangent space of G at e. For each $a \in G$, the *left translation* is defined by

$$L_a : G \to G, \quad x \mapsto ax.$$

It is a diffeomorphism on G.

Definition 2.4 (See [10]). *A vector field X on G is called* left-invariant *if for all $a \in G$,*

$$(T_eL_a)(X(e)) = X(a).$$

Proposition 2.5 (See [10]). *Let $X \in T_eG$, and define a mapping $\tilde{X} : G \to TG$ by*

$$\tilde{X}(f)(p) = X(f \circ L_p), \quad \forall p \in G,$$

for any analytic function f on G. Then $\tilde{X}$ defines a unique left-invariant vector field on G such that $\tilde{X}(e) = X$.

Moreover, any left-invariant vector field is of the form $\tilde{X}$.

Let $\mathcal{L}(G)$ denote the set of all left-invariant vector fields on G. Then every such vector field is uniquely determined by its value at the identity:

$$\phi : \mathcal{L}(G) \to T_eG, \quad \tilde{X} \mapsto \tilde{X}(e),$$

is a vector space isomorphism.

Proposition 2.6 (See [10]). *Let $\tilde{X}, \tilde{Y} \in \mathcal{L}(G)$ be left-invariant vector fields. Then their Lie bracket $[\tilde{X}, \tilde{Y}]$ is also in $\mathcal{L}(G)$.*

Thus, the set $\mathcal{L}(G)$ forms a Lie subalgebra of $\mathfrak{X}(G)$ (the space of smooth vector fields on G). This Lie algebra is called the *Lie algebra of G* and is often denoted by $\mathfrak{g}$. The tangent space T_eG also inherits a Lie algebra structure via

$$[X, Y] := [\tilde{X}, \tilde{Y}](e), \quad \forall X, Y \in T_eG.$$

Therefore, the map $\phi : \mathcal{L}(G) \to T_eG$ is not only a vector space isomorphism but also a Lie algebra isomorphism.

3. RKHS Structure on Smooth Vector Fields (Haar Inner Product)

Let $\mathfrak{X}(G)$ denote the space of smooth vector fields on G. Each $X \in X(G)$ acts on smooth functions $f \in C^\infty(G)$:

$$X(f) : G \to \mathbb{C}, \quad x \mapsto X_x(f),$$

where X_x is the vector at $x \in G$.

Dense Set of Functions

Choose a countable dense set $\{f_i\}_{i\in\mathbb{N}} \subset C^\infty(G)$ with respect to the $L^2(G, \mu)$ norm, where μ is a Haar measure on G. Let $\{w_i\}_{i\in\mathbb{N}}$ be positive weights decaying sufficiently fast to guarantee convergence of the series.

Inner Product

For $X, Y \in \mathfrak{X}(G)$, define the inner product

$$\langle X, Y\rangle_H := \sum_{i=1}^{\infty} w_i \int_G X(f_i)(x)\, \overline{Y(f_i)(x)}\, d\mu(x),$$

with associated norm

$$\|X\|_H^2 = \sum_{i=1}^{\infty} w_i \int_G |X(f_i)(x)|^2\, d\mu(x).$$

This series might not converge, because there is no guarantee that the family of integrals is uniformly bounded,and even if each term is finite, the infinite sum may diverge. Therefore, we introduce positive weights $\{w_i\}_i$

Dense Set of Functions

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Hilbert Space H

Define H as the completion of $X(G)$ under this norm:

$$H := \overline{\{X \in X(G) : \|X\|_H < \infty\}}^{\|\cdot\|_H}.$$

Evaluation Functionals and Riesz Representers

For each $f \in C^\infty(G)$, define the linear functional

$$T_f : H \to L^2(G), \quad T_f(X) := X(f).$$

Since $\{f_i\}$ is dense, for any f we can approximate $f \approx \sum_{i=1}^N c_i f_i$, giving

$$\|X(f)\|_{L^2(G)} \le \sum_{i=1}^N |c_i| \, \|X(f_i)\|_{L^2(G)} \le C_f \|X\|_H.$$

Hence T_f is continuous.

Fix a non-zero vector $\psi \in L^2(G)$ and normalize it if convenient, i.e. assume $\|\psi\|_{L^2(G)} = 1$. For each $f \in C^\infty(G)$ define the scalar-valued linear functional

$$\varepsilon_f : H \to \mathbb{C}, \qquad \varepsilon_f(X) := \langle X(f), \psi \rangle_{L^2(G)}.$$

Using the estimate

$$\|X(f)\|_{L^2(G)} \le C_f \|X\|_H$$

(which follows from the density argument with the $\{f_i\}$ and the definition of $\|\cdot\|_H$), we obtain

$$|\varepsilon_f(X)| = |\langle X(f), \psi \rangle_{L^2(G)}| \le \|X(f)\|_{L^2(G)} \|\psi\|_{L^2(G)} \le C_f \|\psi\|_{L^2(G)} \|X\|_H,$$

so ε_f is a bounded (continuous) linear functional on H. By the Riesz representation theorem there exists a unique vector $X_f \in H$ such that

$$\varepsilon_f(X) = \langle X, X_f \rangle_H, \qquad \forall X \in H.$$

Scalar Reproducing Kernel

Define a scalar kernel $K : C^\infty(G) \times C^\infty(G) \to \mathbb{C}$ by

$$K(f, g) := \langle X_g, X_f \rangle_H.$$

Then K is Hermitian, and positive definite. Moreover, the reproducing identity (for the scalarized evaluations) reads

$$\langle X(f), \psi \rangle_{L^2(G)} = \varepsilon_f(X) = \langle X, X_f \rangle_H = \langle X, K(\cdot, f) \rangle_H, \qquad \forall X \in H,$$

where we denote $K(\cdot, f) = X_f \in H$.

In particular, the map $f \mapsto X_f$ identifies the space of scalarized evaluation functionals with the reproducing kernel Hilbert space having kernel K.

4. Lie algebra of a Lie group with reproducing kernel

Let $\mathfrak{g} \subset \mathfrak{X}(G)$ be a (possibly infinite-dimensional) Lie subalgebra, and let

$$\mathfrak{g}_H := \overline{\mathfrak{g}}^H$$

be its closure in H. Equip $\mathfrak{g}_H$ with the inner product inherited from H:

$$\langle \tilde{X}, \tilde{Y} \rangle_{\mathfrak{g}} := \langle \tilde{X}, \tilde{Y} \rangle_H, \qquad \tilde{X}, \tilde{Y} \in \mathfrak{g}_H.$$

For each $f \in C^\infty(G)$, the evaluation functional on $\mathfrak{g}_H$ has a (unique) representer, again denoted $\tilde{X}_f \in \mathfrak{g}_H$, with

$$\tilde{X}(f) \;=\; \langle \tilde{X}, \tilde{X}_f \rangle_{\mathfrak{g}}, \qquad \forall\, \tilde{X} \in \mathfrak{g}_H.$$

The induced kernel on $C^\infty(G)$ is

$$K_{\mathfrak{g}}(f,g) \;:=\; \langle \tilde{X}_g, \tilde{X}_f \rangle_{\mathfrak{g}},$$

and $(\mathfrak{g}_H, K_{\mathfrak{g}})$ is a reproducing kernel Hilbert space with reproducing property

$$\tilde{X}(f) \;=\; \langle \tilde{X}, K_{\mathfrak{g}}(\cdot, f) \rangle_{\mathfrak{g}}, \qquad \forall\, \tilde{X} \in \mathfrak{g}_H.$$

Let G_1 and G_2 be two Lie groups, $\mathfrak{g}_1$, $\mathfrak{g}_2$ their Lie algebras, respectively and ${\mathfrak{g}^1}_{H_1}$, ${\mathfrak{g}^2}_{H_2}$ their RKHS respectively. Let $K_{g^1} : C^\infty(G_1) \times C^\infty(G_1) \longrightarrow \mathbb{C}$ be the kernel function for ${\mathfrak{g}^1}_{H_1}$. If we have $G_2 \subseteq G_1$, then the restriction of K_{g^1} to $C^\infty(G_2) \times C^\infty(G_2)$ is also a kernel function and we can use K_{g^1} to form a reproducing kernel on $\mathfrak{g}_2$. The evaluation function for ${\mathfrak{g}^2}_{H_2}$ will be the restriction of the evaluation function T_f for ${\mathfrak{g}^1}_{H_1}$ on ${\mathfrak{g}^2}_{H_2}$.

Let us consider a smooth linear operator ϕ from $C^\infty(G_2)$ to $C^\infty(G_1)$, then we define

$$K^\phi_{g^1} : C^\infty(G_2) \times C^\infty(G_2) \longrightarrow \mathbb{C}$$

denote the function given by

$$K^\phi_{g^1}(f,g) = K_{g^1}(\phi(f), \phi(g)) \quad \text{with } f, g \in C^\infty(G_2).$$

Theorem 4.1. *If we consider $\phi : C^\infty(G_2) \longrightarrow C^\infty(G_1)$ linear and smooth, and let $K^\phi_{g^1} : C^\infty(G_1) \times C^\infty(G_1) \longrightarrow \mathbb{C}$ be the kernel function for ${\mathfrak{g}^1}_{H_1}$, then $K^\phi_{g^1}$ is a reproducing kernel on $C^\infty(G_2) \times C^\infty(G_2)$. The reproducing kernel space of the kernel $K^\phi_{g^1}$ is a Lie subalgebra of $\mathfrak{g}_2$ and will be denoted by*

$$\mathfrak{g}^\phi_1 = \{\tilde{X} \circ \phi : \tilde{X} \in {\mathfrak{g}^1}_{H_1}\}.$$

For $\overline{X} \in \mathfrak{g}^\phi_1$, we have

$$\|\overline{X}\|_{\mathfrak{g}^\phi_1} = \min\Big\{ \|\tilde{X}\|_{{\mathfrak{g}^1}_{H_1}} \;;\; \overline{X} = \tilde{X} \circ \phi \Big\}.$$

Proof:

Let us consider $f_1, ..., f_p \in C^\infty(G_1)$; $g_1, ..., g_n \in C^\infty(G_2)$, let $c_1, ..., c_n \in \mathbb{C}$ and let $\{f_1, ..., f_p\} = \{\phi(g_1), ..., \phi(g_n)\}$, such that $p \leq n$.

Let $A_k = \{i : \phi(g_i) = f_k\}$ and let $b_k = \sum_{i \in A_k} c_i$. Then,

$$\begin{aligned}\sum_{i,j=1}^{n} \overline{c_i} c_j K_{g^1}(\phi(g_i), \phi(g_j)) &= \sum_{k,l=1}^{p} \sum_{i \in A_k} \sum_{j \in A_l} \overline{c_i} c_j K_{g^1}(f_k, f_l) \\ &= \sum_{k,l=1}^{p} \overline{b_k} b_l K_{g^1}(f_k, f_l) \\ &\geq 0.\end{aligned}$$

Therefore, $K_{g^1}^{\phi}$ is a reproducing kernel on $C^\infty(G_2) \times C^\infty(G_2)$.

Let $\tilde{X} \in \mathfrak{g^1}_{H_1}$ and $f_1, f_2 \in C^\infty(G_1)$ with $\|\tilde{X}\|_{\mathfrak{g^1}_{H_1}} = c$, then $\tilde{X}(f_1)\overline{\tilde{X}(f_2)} \leq c^2 K_{g^1}(f_1, f_2)$. Since we have this inequality, we see that

$$\tilde{X} \circ \phi(g_1)\overline{\tilde{X} \circ \phi(g_2)} \leq c^2 K_{g^1}(\phi(g_1), \phi(g_2)).$$

Hence, $\tilde{X} \circ \phi \in \mathfrak{g}_1^{\phi}$ with $\|\tilde{X} \circ \phi\| \leq c$. That means there exists a contractive linear map

$U_\phi : \mathfrak{g^1}_{H_1} \longrightarrow \mathfrak{g}_1^{\phi}$ given by $U_\phi(\tilde{X}) = \tilde{X} \circ \phi$. (see [18] theorem 3.11 p.44)

Let us set for $g \in C^\infty(G_2)$, $\zeta_g(.) = K_{g^1}(\phi(.), \phi(g))$, the kernel function for $\mathfrak{g}_1^{\phi}$. For $c_1, ..., c_n \in \mathbb{C}$, if $\overline{X} = \sum_i c_i \zeta_g$, then,

$$\|\overline{X}\|_{\mathfrak{g}_1^{\phi}} = \|\textstyle\sum_i c_i K^1_{\phi(g)}\|_{\mathfrak{g^1}_{H_1}}.$$

That means there exists an isometry, $V : \mathfrak{g}_1^{\phi} \longrightarrow \mathfrak{g^1}_{H_1}$, satisfying

$$V(\zeta_g) = K^1_{\phi(g)},$$

then $U_\phi \circ V$ is the identity on $\mathfrak{g}_1^{\phi}$. Hence, for any $\overline{X} \in \mathfrak{g}_1^{\phi}$, $\tilde{X} = V(\overline{X})$ satisfies $\overline{X} = \tilde{X} \circ \phi$ with $\|\overline{X}\|_{\mathfrak{g}_1^{\phi}} = \|\tilde{X}\|_{\mathfrak{g^1}_{H_1}}$.

Let us prove that $\mathfrak{g}_1^{\phi}$ is a Lie subalgebra of $\mathfrak{g}_2$. Since $\mathfrak{g}_1^{\phi}$ is a subspace of $\mathfrak{g}_2$, we have to prove that for any $\overline{X}, \overline{Y} \in \mathfrak{g}_1^{\phi}$, $[\overline{X}, \overline{Y}] \in \mathfrak{g}_1^{\phi}$.

Let $a \in G_2$, e_2 be the identity element of the Lie group G_2, $A(a)$ be the set of analytic functions at points a and $\varphi \in A(a)$. We have:

$$\begin{aligned}
[dL(a)][\overline{X},\overline{Y}]_{e_2}\varphi &= [\overline{X},\overline{Y}]_{e_2}(\varphi \circ L_a) \\
&= \overline{X}_{e_2}(\overline{Y}(\varphi \circ L_a)) - \overline{Y}_{e_2}(\overline{X}(\varphi \circ L_a)) \\
&= \overline{X}_{e_2}(\overline{Y}_\varphi \circ L_a) - \overline{Y}_{e_2}(\overline{X}_\varphi \circ L_a) \\
&= dL(a)\overline{X}_{e_2}(\overline{Y}_\varphi) - dL(a)\overline{Y}_{e_2}(\overline{X}_\varphi) \\
&= \overline{X}_a(\overline{Y}_\varphi) - \overline{Y}_a(\overline{X}_\varphi) \\
&= [\overline{X},\overline{Y}]_a\varphi
\end{aligned}$$

so, $[dL(a)][\overline{X},\overline{Y}]_{e_2} = [\overline{X},\overline{Y}]_a$. We proved that the reproducing kernel space $\mathfrak{g}_1^\phi$ is a Lie subalgebra of $\mathfrak{g}_2$.

□

4.1. Inversion formula in vector fields with a reproducing kernel

Let $\mathfrak{g} \subset \mathfrak{X}(G)$ be an infinite-dimensional Lie algebra. Consider the reproducing kernel Hilbert space $\mathfrak{g}_H$ with reproducing kernel $K_\mathfrak{g}$. Let $\mathcal{F}(C^\infty(G))$ denote the set of complex-valued functions on $C^\infty(G)$, the space of all infinitely differentiable complex functions on G.

Define a map

$$l : C^\infty(G) \to \mathfrak{g}_H, \quad f \mapsto l(f) = l_f.$$

We also consider the linear map

$$J : \mathfrak{g}_H \to \mathcal{F}(C^\infty(G)), \quad \tilde{X} \mapsto \hat{X} = J\tilde{X},$$

defined by

$$\hat{X}(f) = (J\tilde{X})(f) := \langle \tilde{X}, l_f \rangle_\mathfrak{g}, \quad \forall f \in C^\infty(G),\ \tilde{X} \in \mathfrak{g}_H.$$

Next, define the kernel K for all $f, g \in C^\infty(G)$ by

$$K(f,g) := \langle l_g, l_f \rangle_\mathfrak{g} = J(l_g)(f).$$

Since the inner product $\langle \cdot, \cdot \rangle_\mathfrak{g}$ is sesquilinear, Hermitian, and positive definite, the kernel K is positive definite. By the Aronszajn theorem, there exists a unique Hilbert space $\mathcal{H}_K$ with reproducing kernel K such that, for all $f \in C^\infty(G)$,

$$\hat{X}(f) = \langle \hat{X}, K(\cdot, f) \rangle_{\mathcal{H}_K}, \quad \forall \hat{X} \in \mathcal{H}_K.$$

Let $R(J)$ denote the range of J. We define a norm on $R(J)$ by

$$\|\hat{X}\|_{R(J)} := \inf\{\|\tilde{X}\|_{\mathfrak{g}} : \hat{X} = J\tilde{X}\}.$$

Then $R(J)$ is a Hilbert space satisfying the same properties as $\mathcal{H}_K$ (see [18], Theorem 1, p.21). By uniqueness of reproducing kernel Hilbert spaces, we have $R(J) = \mathcal{H}_K$, and for all $\tilde{X} \in \mathfrak{g}$ and $f \in C^\infty(G)$,

$$(J\tilde{X})(f) = \hat{X}(f) = \langle \hat{X}, K(\cdot, f)\rangle_{\mathcal{H}_K}.$$

Theorem 4.2. *If we consider for the Hilbert space $\mathfrak{g}_H$, We consider $\{\tilde{X}_i\}_{i\in\mathbb{N}}$ be an orthonormal basis of $\mathfrak{g}$ under the scalar product inherited from H, $\hat{X} \in \mathcal{H}_K$, $f \in C^\infty(G)$ and $\hat{X}_i(f) = \langle \tilde{X}_i, l_f\rangle_{\mathfrak{g}}$ for all i. Then,*

1) *For any $f, g \in C^\infty(G)$, $K(f,g) = \sum_{i=1}^{\infty} \tilde{X}_i(f)\overline{\tilde{X}_i(g)}$ and $\|\hat{X}\|_{\mathcal{H}_K} \leq \|\langle \hat{X}, \overline{l(\cdot)}\rangle_{\mathcal{H}_K}\|_{\mathfrak{g}}$.*

2) *Furthermore, if we assume that $\{l_f, f \in C^\infty(G)\}$ is dense in $\mathfrak{g}$ then, $\|\hat{X}\|_{\mathcal{H}_K} = \|\langle \hat{X}, \overline{l(\cdot)}\rangle_{\mathcal{H}_K}\|_{\mathfrak{g}}$ and there exists a unique $\tilde{X}^*$ in $\mathfrak{g}$ such that:*

$$\tilde{X}^* = \langle \hat{X}, \overline{l(\cdot)}\rangle_{\mathcal{H}_K} = \textstyle\sum_{i=1}^{\infty} \langle \hat{X}, \langle \tilde{X}_i, l(\cdot)\rangle_{\mathfrak{g}}\rangle_{\mathcal{H}_K} \tilde{X}_i.$$

Proof :

1) Let us consider $f, g \in C^\infty(G)$, and for all i, $\langle K_f, \hat{X}_i\rangle_{\mathcal{H}_K} = \overline{\langle \hat{X}_i, K_f\rangle_{\mathcal{H}_K}} = \overline{\hat{X}_i(f)}$. By the Parseval identity :

$$\begin{aligned} K(f,g) = \langle K_g, K_f\rangle_{\mathcal{H}_K} &= \sum_{i=1}^{\infty} \langle K_g, \hat{X}_i\rangle_{\mathcal{H}_K} \overline{\langle K_f, \hat{X}_i\rangle_{\mathcal{H}_K}} \\ &= \sum_{i=1}^{\infty} \hat{X}_i(f)\overline{\hat{X}_i(g)}. \end{aligned}$$

$\hat{X}_i(f) = \langle \tilde{X}_i, l_f\rangle_{\mathfrak{g}}$, then $l_f = \sum_{i=1}^{\infty} \langle l_f, \tilde{X}_i\rangle_{\mathfrak{g}} \tilde{X}_i = \sum_{i=1}^{\infty} \overline{\hat{X}_i(f)} \tilde{X}_i$.

Now, if we set $\bar{l}_f = \sum_{i=1}^{\infty} \hat{X}_i(f)\tilde{X}_i$, we have

$$\overline{l(\cdot)} = \sum_{i=1}^{\infty} \hat{X}_i(\cdot)\tilde{X}_i. \tag{1}$$

Let $\hat{X} \in \mathcal{H}_K$ then, $\langle \hat{X}, \overline{l(\cdot)} \rangle_{\mathcal{H}_K} = \sum_{i=1}^{\infty} \langle \hat{X}, \hat{X}_i(\cdot) \rangle_{\mathcal{H}_K} \tilde{X}_i$ and $\langle \hat{X}, \overline{l(\cdot)} \rangle_{\mathcal{H}_K} \in \mathfrak{g}$. Let $f \in C^{\infty}(G)$, since

$$\begin{aligned}\langle l_f, l(\cdot) \rangle_{\mathfrak{g}} &= \langle \sum_{i=1}^{\infty} \overline{\hat{X}_i(f)} \tilde{X}_i, \sum_{i=1}^{\infty} \overline{\hat{X}_i(\cdot)} \tilde{X}_i \rangle_{\mathfrak{g}} \\ &= \sum_{i=1}^{\infty} \overline{\hat{X}_i(f)} \hat{X}_i(\cdot),\end{aligned}$$

then,

$$\begin{aligned}\langle \hat{X}, \langle l_f, l(\cdot) \rangle_{\mathfrak{g}} \rangle_{\mathcal{H}_K} &= \langle \hat{X}, \sum_{i=1}^{\infty} \overline{\hat{X}_i(f)} \hat{X}_i(\cdot) \rangle_{\mathcal{H}_K} \\ &= \sum_{i=1}^{\infty} \hat{X}_i(f) \langle \hat{X}, \hat{X}_i(\cdot) \rangle_{\mathcal{H}_K},\end{aligned}$$

$$\begin{aligned}\langle \langle \hat{X}, \overline{l(\cdot)} \rangle_{\mathcal{H}_K}, l_f \rangle_{\mathfrak{g}} &= \langle \sum_{i=1}^{\infty} \langle \hat{X}, \hat{X}_i(\cdot) \rangle_{\mathcal{H}_K} \tilde{X}_i, l_f \rangle_{\mathfrak{g}} \\ &= \langle \sum_{i=1}^{\infty} \langle \hat{X}, \hat{X}_i(\cdot) \rangle_{\mathcal{H}_K} \tilde{X}_i, \sum_{i=1}^{\infty} \overline{\hat{X}_i(f)} \tilde{X}_i \rangle_{\mathfrak{g}} \\ &= \sum_{i=1}^{\infty} \langle \hat{X}, \hat{X}_i(\cdot) \rangle_{\mathcal{H}_K} \hat{X}_i(f)\end{aligned}$$

So, $\langle \hat{X}, \langle l_f, l(\cdot) \rangle_{\mathfrak{g}} \rangle_{\mathcal{H}_K} = \langle \langle \hat{X}, \overline{l(\cdot)} \rangle_{\mathcal{H}_K}, l_f \rangle_{\mathfrak{g}}$.
The foregoing assumptions and the above equality lead us to:

$$\begin{aligned}\hat{X}(f) &= \langle \hat{X}, K(\cdot, f) \rangle_{\mathcal{H}_K} \\ &= \langle \hat{X}, \langle l_f, l(\cdot) \rangle_{\mathfrak{g}} \rangle_{\mathcal{H}_K} \\ &= \langle \langle \hat{X}, \overline{l(\cdot)} \rangle_{\mathcal{H}_K}, l_f \rangle_{\mathfrak{g}}\end{aligned}$$

Therefore,

$$\hat{X} = J \langle \hat{X}, \overline{l(\cdot)} \rangle_{\mathcal{H}_K}, \|\hat{X}\|_{\mathcal{H}_K} \leq \| \langle \hat{X}, \overline{l(\cdot)} \rangle_{\mathcal{H}_K} \|_{\mathfrak{g}}. \tag{2}$$

2) For some $\tilde{X}_0 \in \mathfrak{g}$, for $i = 1, 2.., n$ and using (1),

$$\begin{aligned}\langle \tilde{X}_0, \langle \hat{X}, \overline{l(\cdot)} \rangle_{\mathcal{H}_K} \rangle_{\mathfrak{g}} &= \langle \tilde{X}_0, \sum_{i=1}^{\infty} \langle \hat{X}, \hat{X}_i(\cdot) \rangle_{\mathcal{H}_K} \tilde{X}_i \rangle_{\mathfrak{g}} \\ &= \sum_{i=1}^{\infty} \overline{\langle \hat{X}, \hat{X}_i(\cdot) \rangle}_{\mathcal{H}_K} \langle \tilde{X}_0, \tilde{X}_i \rangle_{\mathfrak{g}}.\end{aligned}$$

We also have :

$$\begin{aligned}\langle \tilde{X}_0, l(\cdot) \rangle_{\mathfrak{g}} &= \langle \sum_{i=1}^{\infty} \langle \tilde{X}_0, \tilde{X}_i \rangle_{\mathfrak{g}} \tilde{X}_i, l(\cdot) \rangle_{\mathfrak{g}} \\ &= \langle \sum_{i=1}^{\infty} \langle \tilde{X}_0, \tilde{X}_i \rangle_{\mathfrak{g}} \tilde{X}_i, \sum_{i=1}^{\infty} \overline{\hat{X}_i(\cdot)} \tilde{X}_i \rangle_{\mathfrak{g}} \\ &= \sum_{i=1}^{\infty} \langle \tilde{X}_0, \tilde{X}_i \rangle_{\mathfrak{g}} \hat{X}_i(\cdot).\end{aligned}$$

Then,

$$\begin{aligned}\langle \langle \tilde{X}_0, l(\cdot) \rangle_{\mathfrak{g}}, \hat{X} \rangle_{\mathcal{H}_K} &= \sum_{i=1}^{\infty} \langle \hat{X}_i(\cdot), \hat{X} \rangle_{\mathcal{H}_K} \langle \tilde{X}_0, \tilde{X}_i \rangle_{\mathfrak{g}} \\ &= \sum_{i=1}^{\infty} \overline{\langle \hat{X}, \hat{X}_i(\cdot) \rangle}_{\mathcal{H}_K} \langle \tilde{X}_0, \tilde{X}_i \rangle_{\mathfrak{g}}.\end{aligned}$$

We obtain finally :

$$\langle \tilde{X}_0, \langle \hat{X}, \overline{l(\cdot)} \rangle_{\mathcal{H}_K} \rangle_{\mathfrak{g}} = \langle \langle \tilde{X}_0, l(\cdot) \rangle_{\mathfrak{g}}, \hat{X} \rangle_{\mathcal{H}_K}. \tag{3}$$

If $\tilde{X}_0 \in Ker(J)$, then we obtain $\langle \tilde{X}_0, l(\cdot) \rangle_{\mathfrak{g}} = L(\tilde{X}_0)(\cdot) = 0$. We get in (3) that $\langle \tilde{X}_0, \langle \hat{X}, \overline{l(\cdot)} \rangle_{\mathcal{H}_K} \rangle_{\mathfrak{g}} = 0$ and $\langle \hat{X}, \overline{l(\cdot)} \rangle_{\mathcal{H}_K} \in [Ker(J)]^{\perp}$.
If $\{l_f, f \in C^{\infty}(G)\}$ is dense in $\mathfrak{g}$, then $[Ker(L)]^{\perp} = \mathfrak{g}$, which implies that J is an isometry between $[Ker(J)]^{\perp}$ and $R(J)$, then there exists a unique $\tilde{X}^* \in [Ker(J)]^{\perp}$ such that, from (2):

$$\tilde{X}^* = J^{-1} \hat{X} = \langle \hat{X}, \overline{l(\cdot)} \rangle_{\mathcal{H}_K} \text{ and } \|\hat{X}\|_{\mathcal{H}_K} = \|\tilde{X}^*\|_{\mathfrak{g}} = \|\langle \hat{X}, \overline{l(\cdot)} \rangle_{\mathcal{H}_K}\|_{\mathfrak{g}}.$$

For the adjoint J^* of the isometry J between $[Ker(J)]^{\perp}$ and $\mathcal{H}_K$, we have $J^* = J^{-1}$ hence, we obtain :

$$\begin{aligned} J^{-1}\hat{X} = \tilde{X}^* &= \sum_{i=1}^{\infty} \langle \tilde{X}^*, \tilde{X}_i \rangle_{\mathfrak{g}} \tilde{X}_i \\ &= \sum_{i=1}^{\infty} \langle \hat{X}, J\tilde{X}_i \rangle_{\mathcal{H}_K} \tilde{X}_i \\ &= \sum_{i=1}^{\infty} \langle \hat{X}, \langle \tilde{X}_i, l(\cdot) \rangle_{\mathfrak{g}} \rangle_{\mathcal{H}_K} \tilde{X}_i. \end{aligned}$$

□

4.2. Transform of Reproducing Kernel defined in a vector field

We consider the kernel K defined on $C^{\infty}(G) \times C^{\infty}(G)$ by $K(f,g) = \langle l_g, l_f \rangle_{\mathfrak{g}}$ and assume that $\{ l_f, f \in C^{\infty}(G)\}$ is dense in $\mathfrak{g}$.
For any $f, g \in C^{\infty}(G)$ and $\hat{X} \in \mathcal{H}_K$, if $\hat{X}(f) = (J\tilde{X})(f) = \langle \tilde{X}, l_f \rangle_{\mathfrak{g}}$, then there exists a unique $\tilde{X}^* \in [Ker(J)]^{\perp}$ such that:

$$\hat{X}(f) = (J\tilde{X}^*)(f) = \langle \tilde{X}^*, l_f \rangle_{\mathfrak{g}}, \|\tilde{X}^*\|_{\mathfrak{g}} = \|\hat{X}\|_{\mathcal{H}_K}. \tag{4}$$

Secondly, we consider $\mathfrak{g}$ the Lie algebra of the group G and a representation π of $\mathfrak{g}$ on $\mathfrak{g}$ and define the following positive kernel with $x \in \mathfrak{g}$, that is: $K_{\pi_x}(f,g)$ on $C^{\infty}(G) \times C^{\infty}(G)$ defined by :

$$K_{\pi_x}(f,g) = \langle \pi_x l_g, \pi_x l_f \rangle_{\mathfrak{g}} \text{ on } C^{\infty}(G) \times C^{\infty}(G).$$

For all $x \in \mathfrak{g}, K_{\pi_x}$ is positive definite kernel, then by the theorem of Aronszajn, there exists a reproducing kernel space that will be denoted by $H_{K_{\pi_x}} = \pi_x(\mathfrak{g})$, whose kernel is K_{π_x}.

Let us consider the linear transform defined by the following map

$$\begin{array}{rccl} \pi_x \circ L : & \mathfrak{g} & \longrightarrow & H_{K_{\pi_x}} \\ & \tilde{X}^* & \longmapsto & \pi_x \circ J(\tilde{X}^*) = \pi_x(\hat{X}) = \langle \tilde{X}^*, \pi_x l(\cdot) \rangle_{\mathfrak{g}} \end{array} \tag{5}$$

Let us set $\pi_x(\hat{X}) = \hat{X}_{\pi_x}$, for all $\hat{X} \in \mathcal{H}_K$ such that for all $f \in C^{\infty}(G)$, $\pi_x \circ J(\tilde{X}^*)(f) = \pi_x(\hat{X})(f) = \hat{X}_{\pi_x}(f) = \langle \tilde{X}^*, \pi_x l_f \rangle_{\mathfrak{g}}$.

Theorem 4.3. *Taking $x \in \mathfrak{g}$, we consider the mapping $\pi_x : \hat{X} \longrightarrow \hat{X}_{\pi_x}$, from $\mathcal{H}_K$ onto $H_{K_{\pi_x}}$. Then we have :*

$$\|\hat{X}\|_{\mathcal{H}_K} \geq \|\hat{X}_{\pi_x}\|_{H_{K_{\pi_x}}}.$$

Note that we get the equality if and only if π_x is one-to-one, that is, if $\{\ \pi_x l_f, f \in C^\infty(G)\}$ is dense in $\mathfrak{g}$.

In particular,

$$K(f,f) \geq \|K_{\pi_x}(\cdot, f)\|^2_{H_{K_{\pi_x}}} \text{ for all } f \in C^\infty(G).$$

Proof:

Let us consider $f \in C^\infty(G)$, $\tilde{X}^* \in \mathfrak{g}$, $x \in \mathfrak{g}$, $\hat{X}_{\pi_x}(f) = \langle \tilde{X}^*, \pi_x l_f\rangle_{\mathfrak{g}}$, then $\|\hat{X}_{\pi_x}\|_{H_{K_{\pi_x}}} \leq \|\tilde{X}^*\|_{\mathfrak{g}}$ and since $\|\hat{X}\|_{\mathcal{H}_K} = \|\tilde{X}^*\|_{\mathfrak{g}}$.
we have

$$\|\hat{X}\|_{\mathcal{H}_K} \geq \|\hat{X}_{\pi_x}\|_{H_{K_{\pi_x}}}. \tag{6}$$

If π_x is one-to-one, then $\{\ \pi_x l_f, f \in C^\infty(G)\}$ is dense in $\mathfrak{g}$ and we obtain the equality.

We assume that π_x is one-to-one and $\{\ \pi_x l_f, f \in C^\infty(G)\}$ is dense in $\mathfrak{g}$, then the map considered in (5) is an isometry. When we consider the inversion formula for π_x, for any $\hat{X}_{\pi_x} \in H_{K_{\pi_x}}$, we take the unique $\hat{X}^* \in \mathcal{H}_K$ such that

$$\pi_x \hat{X}^* = \hat{X}_{\pi_x} \text{ and } \|\hat{X}^*\|_{\mathcal{H}_K} = \|\hat{X}_{\pi_x}\|_{H_{K_{\pi_x}}}.$$

Then,

$$\hat{X}_{\pi_x}(f) = \langle \hat{X}^*, \pi_x l_f\rangle_{\mathfrak{g}}, \hat{X}^* \in [Ker(\pi_x)]^\perp$$

and

$$\hat{X}^*(f) = \langle \tilde{X}^*, l_f\rangle_{\mathfrak{g}}.$$

Note that the space $[Ker(\pi_x)]^\perp$ denotes the orthogonal complement in $\mathfrak{g}$ of the null space of π_x. Then, we have $\hat{X}^*(f) =< \tilde{X}^*, l_f >_{\mathfrak{g}} = \langle \hat{X}_{\pi_x}, \pi_x K_f\rangle_{H_{K_{\pi_x}}}$, using the isometry $\pi_x \circ J$ from $\mathfrak{g}$ onto $H_{K_{\pi_x}}$. Then for all $f \in C^\infty(G)$,

$$K_f(f) = \langle l_f, l_f\rangle_{\mathfrak{g}}$$

$$= \langle Jl_f, Jl_f \rangle_{\mathcal{H}_K}$$
$$= \langle K_f, K_f \rangle_{\mathcal{H}_K}$$
$$= \langle \pi_x K_f, \pi_x K_f \rangle_{H_{K_{\pi_x}}}$$
$$= \|K_{\pi_x}(\cdot, f)\|^2_{H_{K_{\pi_x}}}$$

Since, $K_f(f) = K(f, f) = \|K(\cdot, f)\|^2_{\mathcal{H}_K}$, then $\|K(\cdot, f)\|^2_{\mathcal{H}_K} = \|K_{\pi_x}(\cdot, f)\|^2_{H_{K_{\pi_x}}}$ and

$$\|K(\cdot, f)\|_{\mathcal{H}_K} = \|K_{\pi_x}(\cdot, f)\|_{H_{K_{\pi_x}}}. \tag{7}$$

In the case where $\{\pi_x l_f, f \in C^\infty(G)\}$ is not dense in $\mathfrak{g}$ or π_x is not one-to-one then, $K(f, f) = \|K(\cdot, f)\|^2_{\mathcal{H}_K} \geq \|K_{\pi_x}(\cdot, f)\|^2_{H_{K_{\pi_x}}}$. (using (6),(7))

□

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