Journal of Contemporary Applied Mathematics
V. 16, No 1, 2026, January-June
ISSN 2222-5498, E-ISSN 3006-3183
https://doi.org/10.62476/jcam.161.1

Frankl's problem for a three-dimensional equation of mixed type with a singular coefficient

Kamoliddin T. Karimov, Asrorjon M. Shokirov

Abstract. The paper studies the Frankl problem for a three - dimensional equation of mixed type with a singular coefficient in a mixed domain consisting of a quarter of a cylinder and two rectangular prisms. The unique solvability of the problem in the class of regular solutions is proved. The Fourier method based on separation of variables is used. The eigenfunctions of the problem in elliptic and hyperbolic domains are constructed. The completeness of the system of eigenfunctions is investigated. Based on the completeness property of systems of eigenfunctions, the uniqueness theorem is proved, and the solution to the problem under study is constructed as the sum of a double series. When substantiating the uniform convergence of the constructed series, asymptotic estimates of the Bessel functions of the real and imaginary arguments were used. Based on them, estimates were obtained for each member of the series, which made it possible to prove the convergence of the obtained series and its derivatives up to the second order inclusive, as well as the existence theorem in the class of regular solutions.

Key Words and Phrases: Frankl's problem, mixed type equations, singular coefficient, Bessel function, modified Bessel function, MacDonald's function, Gauss hypergeometric function, Bessel operator.

2010 Mathematics Subject Classifications: 35M10, 35M12.

1. Introduction

The first results concerning Frankl's problem for the Lavrentiev–Bitsadze equation $u_{xx} + (\operatorname{sgn} y)\, u_{yy} = 0$ and for the Chaplygin equation $K(y)\, u_{xx} + u_{yy} = 0,\ K(0) = 0,\ K'(y) > 0$ were obtained by A. V. Bitsadze [1], [2].

In the works of Yu. V. Devingtal [3]–[5] the Frankl problem for the Chaplygin equation was studied for $K(y) \in C^1\left(\bar{D}\right),\ K(-y) = -K(y)$, and E. I. Moiseev in the work [6] by the method of spectral analysis constructed a solution to the Frankl problem for the equation

$$u_{xx} + \operatorname{sgn} y \cdot u_{yy} - \mu^2 \operatorname{sgn}(x+y)\, u = 0, \;\; \mu = const.$$

Also by his students [7]–[11] studied the problems of completeness and basis of the system of eigenfunctions in the ellipticity domain of the modified Frankl problem with nonlocal conditions of the first and second kind.

After these works, interest in studying the Frankl problem increased. Note that such problems were the object of study in [12]–[15]. In these works, some generalizations of the Frankl problem or analogs of the Frankl problem are considered in special areas.

In the work [16], the basis property of the system of eigenfunctions, corresponding to a problem with a spectral parameter in the boundary condition is studied. In the work [17], the eigenvalues of Fredholm type limit integral equations in the space of Bohr almost periodic functions are studied.

The works listed devoted to this problem mainly concern two-dimensional problems. Three-dimensional problems are considered in works [18], [19], while Frankl's problem for mixed-type three-dimensional equations with singular coefficients remains poorly studied.

In this work, the Fourier spectral method is used on the basis of the separation of variables. The Fourier series method is widely used in the development of the modern theory of partial differential equations (see, for example, [20]–[25]).

2. Statement of the problem

In this paper, we consider the following equation

$$\operatorname{sgn}(x+y)\left[U_{xx} + \operatorname{sgn} y \cdot U_{yy}\right] + U_{zz} + \frac{2\gamma}{z} U_z = 0 \tag{1}$$

in the domain $D = \Omega \times (0, c)$, where $\gamma = const \in R$, and $\gamma \in (-\infty, 1/2)$, and Ω is a finite simply connected domain of the plane xOy, bounded by the arc $\bar{\sigma}_0 = \left\{(x,y) : x^2 + y^2 = 1,\ x \geq 0,\ y \geq 0\right\}$ and the segments $\overline{MM^*} = \{(x,y) : x = 0,\ -1 \leq y \leq 1\}$, $\overline{M^*P} = \{(x,y) : x - y = 1, 0 \leq x \leq 1\}$.

Let us introduce the following notations: $\Omega_0 = \Omega \cap (y > 0)$, $\Omega_1 = \Omega \cap \{(x,y) : y < 0,\ x + y > 0\}$, $\Omega_2 = \Omega \cap \{y < 0,\ x + y < 0\}$, $D_j = \Omega_j \times (0, c)$, $j = \overline{0, 2}$, $OP = \Omega \cap (y = 0)$, $OQ = \{(x,y) : x + y = 0,\ 0 < x < 1/2\}$, $OM = \{(x,y) : x = 0, 0 < y < 1\}$, $OM^* = MM^* \backslash \overline{OM}$, $O(0,0)$, $M(0,1)$, $M^*(0,-1)$.

In the domain D, the equation (1) belongs to the mixed type, namely: in the domain D_0 to the elliptic type, and in the domains D_1 and D_2 to the hyperbolic type. The rectangle $OP \times (0, c)$ is the plane of change of type, and $z = 0$ is the planes of singularity of the coefficient of the equation. In addition, when passing from the domain D_1 and D_2 (and vice versa) through the rectangle $OQ \times (0, c)$, the coefficients of the equation at the second derivatives have a discontinuity of the first kind.

By a regular solution in the domain D of the equation (1), we mean a function $U(x, y, z)$ from the class of functions

$$C\left(\bar{D}\right) \cap C^1\left(\left(D \cup \{x = 0\}\right) \backslash \left(\bar{D}_1 \cap \bar{D}_2\right)\right) \cap C^{2,2,2}_{x,y,z}\left(D_0 \cup D_1 \cup D_2\right). \tag{2}$$

In the domain D, we study the following problem for equation (1).

Problem $\mathbf{F}^{(1)}$. Find a regular solution of the equation (1) in domain D satisfying the conditions

$$U(x, y, z) = 0, \ \ (x, y) \in \bar{\sigma}_0, \ \ z \in (0, c), \tag{3}$$

$$U(0, y, z) = 0, \ \ y \in [-1, 1], \ \ z \in (0, c), \tag{4}$$

$$U_x(0, y, z) + q_1 U_x(0, -y, z) = 0, \ \ y \in (-1, 0), \ \ z \in [0, c], \tag{5}$$

$$U(x, y, 0) = f_1(x, y), \quad U(x, y, c) = f_2(x, y), \ \ (x, y) \in \bar{\Omega}, \tag{6}$$

where $f_1(x, y)$ and $f_2(x, y)$ are given functions, and q_1 equals 1 or -1.

Before moving on to the study of the problem considered, let us present some known facts that we will need below.

Two-dimensional Cauchy–Goursat problem. Find a solution of the following equation

$$u_{xx} - u_{yy} + \lambda u = 0, \quad (x, y) \in \Omega_1,$$

that is regular in the domain Ω_1 and satisfying the conditions

$$u_y(x, 0) = \nu_1(x), \quad x \in (0, 1), \quad u(x, -x) = \psi_1(2x), \quad x \in (0, 1/2),$$

where $\nu_1(x)$ and $\psi_1(x)$ are given functions, such that $\nu_1(x) \in C^2(0, 1)$ and it can have a singularity of order less than one at $x \to 0$ and $x \to 1$, and $\psi_1(x) \in C^1[0, 1] \cap \cap C^{(2,\delta)}(0, 1)$, $\delta > 0$.

Using the results of [26], it is easy to verify that the solution to this problem in the domain Ω_1 exists, is unique, and can be represented in the form

$$u(x, y) = \int_0^{\xi} v(t) J_0\left[\sqrt{\lambda(\xi - t)(\eta - t)}\right] dt + \psi_1(\xi/2) +$$

$$+\psi_1(\eta/2)-\psi_1(0)\,J_0\left(\sqrt{\lambda\xi\eta}\right)-\int\limits_0^{\eta}\psi_1(z/2)\,B_2(0,z;\xi,\eta)dz, \tag{7}$$

where $\xi = x+y$, $\eta = x-y$, $J_\nu(x)$ is the Bessel's function of the first kind of order ν [27],

$$B(t,z;\xi,\eta)=\begin{cases} J_0\left[\sqrt{\lambda(\xi-t)(\eta-t)}\right], & z>\xi,\\ J_0\left[\sqrt{\lambda(\xi-t)(\eta-z)}\right]+J_0\left[\sqrt{\lambda(\xi-z)(\eta-t)}\right], & z<\xi.\end{cases}$$

Using the method of separation of variables, it is easy to prove that the following problems have only trivial solutions:

Problem 1. Find a solution to the equation (1) in the domain D_2 satisfying the homogeneous conditions $U(0,y,z)=0$, $y\in[-1,0]$, $z\in[0,c]$; $U_x(0,y,z)=0$, $y\in(-1,0)$, $z\in(0,c)$; $U(x,y,0)=0$, $U(x,y,c)=0$, $(x,y)\in\Omega_2$.

Problem 2. Find a solution to equation (1) in domain D_1 satisfying the homogeneous conditions $U(x,0,z)=0$, $x\in[0,1]$, $z\in[0,c]$; $U_y(x,0,z)=0$, $x\in(0,1)$, $z\in(0,c)$; $U(x,y,0)=0$, $U(x,y,c)=0$, $(x,y)\in\Omega_1$.

Now let us return to the study of the main problem. It is obvious that the problem $\mathrm{F}^{(1)}$ is a problem of the Frankl problem type [26], [28]. If $q_1=0$, then it follows from (5) that $U_x(0,y,z)=0$, $y\in(-1,0)$, $z\in(0,c)$. Then, taking into account this and (4), as well as the conditions (6) for $f_1(x,y)=f_2(x,y)=0$, based on the triviality of the solution to problem 1 for equation (1) in domain D_2, we have $U(x,y,z)\equiv 0$, $(x,y,z)\in\bar{D}_2$, whence it follows that $U(x,y,z)|_{\overline{OQ}\times[0,c]}=0$. Consequently, in this case the problem $\mathrm{F}^{(1)}$ is equivalent to the Tricomi problem for the equation (1) in the domain $D_0\cup[OP\times(0,c)]\cup D_1$. The Tricomi problem for an equation with singular coefficients was studied in [29], [30].

3. Construction of a non-trivial solution to the problem {(1)–(5)}

We find non-trivial solutions of the equation (1) that satisfy conditions (2)–(5). Separating the variables according to the formula $U(x,y,z)=u(x,y)\,Z(z)$, from the equation (1) and the conditions (2) -(5), we obtain the equation

$$Z''(z)+\frac{2\gamma}{z}Z'(z)-\lambda Z(z)=0,\ \ 0<z<c, \tag{8}$$

and the following spektral problem:

$$u(x,y)\in C\left(\bar{\Omega}\right)\cap C^1\left((\Omega\cup\{x=0\})\setminus\overline{OQ}\right)\cap C^{2,2}_{x,y}\left(\Omega_0\cup\Omega_1\cup\Omega_2\right), \tag{9}$$

$$u_{xx}+(\mathrm{sgn}y)\,u_{yy}+\lambda\mathrm{sgn}\,(x+y)\,u=0,\quad (x,y)\in\Omega_0\cup\Omega_1\cup\Omega_2, \tag{10}$$

$$u(x,y)=0, \quad (x,y)\in\bar{\sigma}_0, \tag{11}$$

$$u(0,y)=0, \quad y\in[-1,1], \tag{12}$$

$$u_x(0,y)+q_1u_x(0,-y)=0, \quad y\in(-1,0). \tag{13}$$

For convenience, we will denote the problem $\{(9)–(13)\}$ by $\mathrm{F}_\lambda^{(1)}$ and solve this problem. In the domain Ω_1, we consider the function $V(x,y)=u(x,y)-u(s,t)$, where $s=-y,\ t=-x$. Obviously, if $(x,y)\in\Omega_1$, then $(s,t)\in\Omega_2$. Taking this into account, as well as the properties of the function $u(x,y)$ and the equality $\lim\limits_{y\to-x-0}u(x,y)=\lim\limits_{y\to-x+0}u(x,y), \quad 0\le x\le 1/2$ that follows from the condition of the problem $\mathrm{F}_\lambda^{(1)}$, it is easy to verify that the function $V(x,y)$ is a solution of the equation (10) in the domain Ω_1, satisfying the condition $V(x,-x)=0$. Then, according to the formula (7), it can be represented as

$$V(x,y)=\int\limits_0^{x+y}V_y(t,0)\,J_0\left[\sqrt{\lambda(\xi-t)(\eta-t)}\right]dt,$$

where $\xi=x+y,\ \ \eta=x-y$.

Assuming $y=0$ in this formula and taking into account the following notations

$$\tau_1(x)=u(x,0), \ \ x\in[0,1], \ \ \nu_1(x)=\lim_{|y|\to0}u_y(x,y), \ \ x\in(0,1), \tag{14}$$

$$\tau_2(y)=u(0,y), \ \ y\in[-1,1], \ \ \nu_2(y)=\lim_{|x|\to0}u_x(x,y), \ \ y\in(-1,1), \tag{15}$$

we obtain

$$\tau_1(x)-\tau_2(-x)=\int\limits_0^x[\nu_1(t)+\nu_2(-t)]\,J_0\left[\sqrt{\lambda}(x-t)\right]dt, \ \ 0\le x\le1. \tag{16}$$

From the equality (16), due to the conditions (12) and (13), it follows that

$$\tau_1(x)=\int\limits_0^x[\nu_1(t)-q_1\nu_2(t)]\,J_0\left[\sqrt{\lambda}(x-t)\right]dt, \ \ 0\le x\le1. \tag{17}$$

(17) is a functional relation between $\tau_1(x)$, $\nu_1(x)$ and $\nu_2(x)$ on the segment $[0,1]$, obtained from the condition that the solution of the problem $\mathrm{F}_\lambda^{(1)}$ is continuous when passing through the lines OQ, and the domain $\Omega_1\cup\Omega_2$ satisfies the equation (10), on the segment $OM*-$ the conditions (12) and (13).

This reduces the problem $F_\lambda^{(1)}$ equivalently to the following generalized eigenvalue problem for the equation (10) in the domain Ω_0 : find the eigenvalues and eigenfunctions of the problem {(10), (11), (12), (17)}.

In the equation (10) in the domain Ω_0, we pass to polar coordinates (r, φ) by the formulas $r = \sqrt{x^2 + y^2}$, $\varphi = arctg\,(y/x)$, and then separate the variables by the formula $u\,(x, y) = R\,(r)\,\Phi\,(\varphi)$. Then the problem splits into two problems with respect to the functions $R\,(r)$ and $\Phi\,(\varphi)$, namely: with respect to the function $R\,(r)$ we obtain an eigenvalue problem

$$r^2 R''\,(r) + rR'\,(r) + \left(\lambda r^2 - \omega^2\right) R\,(r) = 0, \;\; r \in (0, 1)\,, \tag{18}$$

$$R\,(0) = 0, \quad R\,(1) = 0, \tag{19}$$

and with respect to the function $\Phi\,(\varphi)$ — the problem of determining those values of the parameter ω for which non-trivial solutions of the equation exist

$$\Phi''\,(\varphi) + \omega^2 \Phi\,(\varphi) = 0, \;\; 0 < \varphi < \pi/2, \tag{20}$$

satisfying the conditions

$$\Phi\,(\pi/2) = 0, \tag{21}$$

$$R\,(x)\,\Phi\,(0) = \left[\Phi'\,(0) - q_1 \Phi'\,(\pi/2)\right] \int\limits_0^x t^{-1} R\,(t)\, J_0 \left[\sqrt{\lambda}\,(x - t)\right] dt, \quad x \in [0, 1]\,, \tag{22}$$

where ω^2 is the separation constant.

A non-trivial solution of the equation (18), satisfying the first condition from (19), to within a constant factor, is the function

$$R\,(r) = J_\omega\left(\sqrt{\lambda} r\right), \;\; \mathrm{Re}\omega > 0. \tag{23}$$

Substituting the function (23) into the equality (22) and computing the resulting integral using the formula [7]

$$\int\limits_0^u J_p\,(c\xi)\, J_q\,(cu - c\xi)\, \frac{d\xi}{\xi} = \frac{1}{p} J_{p+q}\,(cu)\,, \quad u, \mathrm{Re}p > 0, \;\; \mathrm{Re}q > -1,$$

we find the second boundary condition for determining $\Phi\,(\varphi)$:

$$\Phi'\,(0) - q_1 \Phi'\,(\pi/2) - \omega \Phi\,(0) = 0, \quad \mathrm{Re}\omega > 0. \tag{24}$$

Thus, with respect to the function $\Phi(\varphi)$ we have a generalized eigenvalue problem $\{(20),\ (21),\ (24)\}$. We seek nontrivial solutions to this problem in the form

$$\Phi(\varphi) = a\cos(\omega\varphi) + b\sin(\omega\varphi), \tag{25}$$

where a and b are arbitrary constants.

Satisfying the function (25) with the conditions (21) and (24), we obtain

$$\begin{cases} a\cos(\omega\pi/2) + b\sin(\omega\pi/2) = 0, \\ a\left[1 - q_1\sin(\omega\pi/2)\right] - b\left[1 - q_1\cos(\omega\pi/2)\right] = 0. \end{cases} \tag{26}$$

The system (26) has nontrivial solutions if ω satisfies the equation

$$\sin(\omega\pi/2) + \cos(\omega\pi/2) = q_1.$$

This equation for $|q_1| \le \sqrt{2}$ has a countable number of real roots, and the positive ones are determined by the equalities

$$\omega_{1,n} = 4(n-1) + \frac{1}{2} + \frac{2}{\pi}\arccos\left(\frac{q_1}{\sqrt{2}}\right), \quad -\sqrt{2} \le q_1 \le \sqrt{2};$$

$$\omega_{2,n} = \begin{cases} 4(n-1) + \frac{1}{2} - \frac{2}{\pi}\arccos\left(\frac{q_1}{\sqrt{2}}\right), & 1 < q_1 \le \sqrt{2}; \\ 4n + \frac{1}{2} - \frac{2}{\pi}\arccos\left(\frac{q_1}{\sqrt{2}}\right), & -\sqrt{2} \le q_1 \le 1, \end{cases}$$

where $n \in N$.

Let $q_1 = 1$. Then the eigenvalues $\omega_{1,n} = 4n - 3$ and $\omega_{2,n} = 4n$, in combined form are written as follows:

$$\omega_n = 2n + \left[(-1)^n - 1\right]/2, \ \ n \in N.$$

If $q_1 = -1$, then the eigenvalues $\omega_{1,n} = 4n - 2$ and $\omega_{2,n} = 4n - 1$, in combined form are written as follows:

$$\omega_n = 2n - \left[(-1)^n + 1\right]/2, \ \ n \in N.$$

Substituting into (25) $\omega = \omega_n$ ($q_1 = 1$ or $q_1 = -1$) and taking into account the first equality from (26), we find non-trivial orthonormal solutions to the problem $\{(20),\ (21),\ (24)\}$:

$$\Phi_n(\varphi) = \frac{2}{\sqrt{\pi}}\sin\left[(\pi/2 - \varphi)\,\omega_n\right], \ \ n \in N. \tag{27}$$

Next, substituting $\omega = \omega_n,\ n \in N$ into (23) and applying the second of the conditions (19), we obtain equations for λ :

$$J_{\omega_n}\left(\sqrt{\lambda}\right) = 0, \quad n \in N. \tag{28}$$

Since $\omega_n > 0$, each of the equations (28) has a countable number of real roots. Denoting by α_{nm}-$m-$ th positive root of the equation (28), we obtain the eigenvalues of the problem $\{(18), (19)\}$:

$$\lambda_{nm} = \alpha_{nm}^2, \ \ n, m \in N. \tag{29}$$

The eigenfunctions of this problem corresponding to the eigenvalues (29), orthonormalized with weight r, according to (23), have the form

$$R_{nm}(r) = \sqrt{2} J_{\omega_n}(\alpha_{nm} r) / J_{\omega_n+1}(\alpha_{nm}), \ \ n, m \in N. \tag{30}$$

Substituting (30) and (27) into the equality $u(x, y) = R(r)\Phi(\varphi)$, we obtain non-trivial solutions to the elliptic problem:

$$u_{nm}(x, y) = d_{nm} \sin[(\pi/2 - \varphi)\omega_n] J_{\omega_n}(\alpha_{nm} r), \ \ (x, y) \in \Omega_0, \ \ n, m \in N, \tag{31}$$

where $d_{nm} = \left[2\sqrt{2}\right] / [\sqrt{\pi} J_{\omega_n+1}(\alpha_{nm})]$.

Therefore, the numbers (29) are the eigenvalues of the problem $\mathrm{F}_\lambda^{(1)}$.

Now we will find, in the domain Ω_1, the eigenfunctions of problem $\mathrm{F}_\lambda^{(1)}$ corresponding to the eigenvalues (29). For this aim, following [31], in the domain Ω_1 we introduce new variables $\xi = \sqrt{x^2 - y^2}$, $\eta = x^2/\xi^2$. In the coordinates (ξ, η) the equation (10) takes the form

$$4\eta(1-\eta) u_{\eta\eta} + 4(1/2 - \eta) u_\eta + \xi^2 u_{\xi\xi} + \xi u_\xi + \lambda \xi^2 u = 0.$$

Separating the variables $u(\xi, \eta) = X(\xi) Y(\eta)$, we get

$$X''(\xi) + \frac{1}{\xi} X'(\xi) + \left(\lambda - \rho^2/\xi^2\right) X(\xi) = 0, \ \ \xi \in (0, 1); \tag{32}$$

$$\eta(1-\eta) Y''(\eta) + (1/2 - \eta) Y'(\eta) + \left(\rho^2/4\right) Y(\eta) = 0, \ \ \eta \in (1, +\infty), \tag{33}$$

where ρ^2 is the separation constant.

One can show that the solution of the equation (32), bounded as $\xi \to 0$, is the function $X(\xi) = J_\rho\left(\sqrt{\lambda}\xi\right)$, $\mathrm{Re}\rho \geq 0$.

(33) is the Gauss hypergeometric equation [32]. Its general solution is given by

$$Y(\eta) = c_1 \eta^{\rho/2} F\left(-\frac{\rho}{2}, \frac{1-\rho}{2}, 1-\rho; \frac{1}{\eta}\right) + c_2 \eta^{-\rho/2} F\left(\frac{\rho}{2}, \frac{1+\rho}{2}, 1+\rho; \frac{1}{\eta}\right), \tag{34}$$

where c_1 and c_2 are arbitrary constants.

Using the following formula [32]

$$F\left(a-1/2,a,2a;\eta\right)=\left(\frac{1}{2}+\frac{1}{2}\sqrt{1-\eta}\right)^{1-2a},$$

the function (34) can be rewritten as

$$Y\left(\eta\right)=c_1 2^{-\rho}\left(\frac{x-y}{x+y}\right)^{\rho/2}+c_2 2^{\rho}\left(\frac{x+y}{x-y}\right)^{\rho/2}.$$

From what has been proved above it follows that functions of the form

$$u\left(x,y\right)=J_\rho\left(\sqrt{\lambda\left(x^2-y^2\right)}\right)\left[c_3\left(\frac{x-y}{x+y}\right)^{\rho/2}+c_4\left(\frac{x+y}{x-y}\right)^{\rho/2}\right],\ \ (x,y)\in\Omega_1, \tag{35}$$

where $\mathrm{Re}\rho\geq 0$, and c_3 and c_4 arbitrary constants, are solutions of equation (10) in the domain Ω_1.

Now, we will prove that the numbers (29) are eigenvalues of the problem $\mathrm{F}_\lambda^{(1)}$. In this case, we will use the functions (31). From them, we find

$$\left.\begin{array}{l} u_{nm}\left(x,0\right)=d_{nm}\sin\frac{\omega_n\pi}{2}J_{\omega_n}\left(\alpha_{nm}x\right),\ \ x\in[0,1]; \\ \frac{\partial}{\partial y}u_{nm}\left(x,0\right)=-d_{nm}\omega_n\cos\frac{\omega_n\pi}{2}x^{-1}J_{\omega_n}\left(\alpha_{nm}x\right),\ \ x\in(0,1). \end{array}\right\} \tag{36}$$

As non-trivial functions in the domain Ω_1, we take functions satisfying the equation (10) for $\lambda_{nm}=\alpha_{nm}^2$, $n,m\in N$ in the domain Ω_1 and the conditions (36). According to the formula (35) and the conditions (36), these solutions have the form

$$u_{nm}\left(x,y\right)=\frac{d_{nm}}{2}\left[q_1\left(\frac{x-y}{x+y}\right)^{\omega_n/2}-(-1)^n\sqrt{2-q_1^2}\left(\frac{x+y}{x-y}\right)^{\omega_n/2}\right]\times$$

$$\times J_{\omega_n}\left[\alpha_{nm}\sqrt{x^2-y^2}\right],\ \ (x,y)\in\Omega_1,\ \ n,m\in N. \tag{37}$$

Next, from (31), we find

$$\frac{\partial}{\partial x}u_{nm}\left(0,y\right)=d_{nm}\omega_n y^{-1}J_{\omega_n}\left(\alpha_{nm}y\right),\ \ y\in(0,1).$$

Taking this into account, from the condition (13), we have

$$\frac{\partial}{\partial x}u_{nm}\left(0,y\right)=d_{nm}q_1\omega_n y^{-1}J_{\omega_n}\left(-\alpha_{nm}y\right),\ \ y\in(-1,0). \tag{38}$$

Replacing the variables x, y with $(-y)$, $(-x)$, in the formula (35) respectively, we obtain a set of functions satisfying the equation (10) in the domain Ω_2 :

$$u(x,y) = \left[c_5 \left(\frac{y-x}{y+x} \right)^{\rho/2} + c_6 \left(\frac{y+x}{y-x} \right)^{\rho/2} \right] J_\rho \left[\sqrt{\lambda (y^2 - x^2)} \right], \ (x,y) \in \Omega_2, \tag{39}$$

where c_5, c_6 are arbitrary constants, and $\mathrm{Re}\rho \geq 0$.

Putting in (39) $\rho = \omega_n$, $\lambda = \alpha_{nm}^2$, $c_5 = q_1 d_{nm}/2$, $c_6 = -q_1 d_{nm}/2$, we obtain solutions of the equation (10) for $\lambda = \alpha_{nm}$, $n, m \in N$, satisfying the conditions (38) and $u_{nm}(0, y) = 0$, $y \in (-1, 0)$:

$$u_{nm}(x,y) = \frac{q_1 d_{nm}}{2} \left[\left(\frac{y-x}{y+x} \right)^{\omega_n/2} - \left(\frac{y+x}{y-x} \right)^{\omega_n/2} \right] \times$$
$$\times J_{\omega_n} \left[\alpha_{nm} \sqrt{y^2 - x^2} \right], \ (x,y) \in \Omega_2, \ n, m \in N. \tag{40}$$

Now, using (31), (37) and (40) in the domain Ω, we compose the following system of functions:

$$u_{nm}(x,y) = \begin{cases} d_{nm} \sin \left[\left(\frac{\pi}{2} - \varphi \right) \omega_n \right] J_{\omega_n} (\alpha_{nm} r), \ (x,y) \in \Omega_0; \\ \frac{d_{nm}}{2} \left[q_1 \left(\frac{x-y}{x+y} \right)^{\omega_n/2} - (-1)^n \sqrt{2 - q_1^2} \left(\frac{x+y}{x-y} \right)^{\omega_n/2} \right] \times \\ \times J_{\omega_n} \left[\alpha_{nm} \sqrt{x^2 - y^2} \right], \ (x,y) \in \Omega_1, \ n, m \in N; \\ \frac{q_1 d_{nm}}{2} \left[\left(\frac{y-x}{y+x} \right)^{\omega_n/2} - \left(\frac{y+x}{y-x} \right)^{\omega_n/2} \right] \times \\ \times J_{\omega_n} \left[\alpha_{nm} \sqrt{y^2 - x^2} \right], \ (x,y) \in \Omega_2, \ n, m \in N. \end{cases} \tag{41}$$

From the form and method of construction it follows that each function of this system is non-trivial in the domain Ω and satisfies all the conditions of the problem $\mathrm{F}_\lambda^{(1)}$ for $\lambda = \alpha_{nm}^2$, $n, m \in N$. Therefore, the numbers (29) are eigenvalues, and (41) are eigenfunctions of the problem $\mathrm{F}_\lambda^{(1)}$.

Setting in the equation (8) $\lambda = \alpha_{nm}^2$, we find its general solution [33]:

$$Z_{nm}(z) = c_7 z^{1/2-\gamma} I_{1/2-\gamma}(\alpha_{nm} z) + c_8 z^{1/2-\gamma} K_{1/2-\gamma}(\alpha_{nm} z), \ z \in [0, c], \tag{42}$$

where c_7 and c_8 are arbitrary constants, $I_l(x)$ and $K_l(x)$ are the Bessel function of the imaginary argument, and the Macdonald function of order l [27], respectively.

Then, the functions

$$U_{nm}(x,y,z) = u_{nm}(x,y) Z_{nm}(z), \ n, m \in N,$$

where $Z_{nm}(z)$ and $u_{nm}(x,y)$ are functions defined by equalities (42) and (41), which are continuous and non-trivial in D solutions of equation (1), satisfying conditions (2)–(5).

4. Study of the completeness of the system of eigenfunctions

For $q_1 = 1$ and $q_1 = -1$, we investigate the completeness of the system of eigenfunctions (41) of problem $\mathrm{F}_{\lambda}^{(1)}$ in the domain Ω_0.

a) Let $q_1 = 1$. Then $\omega_n = 2n + [(-1)^n - 1]/2$, $n \in N$.

Theorem 1. *The system of eigenfunctions* $\{u_{nm}(x,y)\}_{n,m=1}^{\infty}$ *is complete in the space* $L_2(\Omega_0)$, *where* $u_{nm}(x,y)$ *is defined by the formula* (41).

The proof of this theorem is based on the following lemma.

Lemma 1. *The system of functions*

$$\{\cos[(4n-3)\varphi]\}_{n=1}^{\infty} \cup \{\sin[(4n\varphi)]\}_{n=1}^{\infty} \tag{43}$$

is complete in the space $L_p(0, \pi/2)$, $p > 1$.

Proof. Let $f(\varphi) \in L_p(0, \pi/2)$, $p > 1$ and

$$\int_0^{\pi/2} f(\varphi)\cos[(4n-3)\varphi]\,d\varphi = 0, \quad \int_0^{\pi/2} f(\varphi)\sin(4n\varphi)\,d\varphi = 0, \; n \in N. \tag{44}$$

Consider the second integral from (44) and rewrite it as

$$0 = \int_0^{\pi/4} f(\varphi)\sin(4n\varphi)\,d\varphi + \int_{\pi/4}^{\pi/2} f(\varphi)\sin(4n\varphi)\,d\varphi, \quad n \in N.$$

Changing the variables by $\varphi = (\pi/2) - \theta$ in the second integral, we have

$$\int_0^{\pi/4} \{f(\varphi) - f[(\pi/2) - \varphi]\}\sin(4n\varphi)\,d\varphi = 0, \quad n \in N.$$

Hence, after changing variables $\theta = 4\varphi$, we get

$$\int_0^{\pi} \left[f\left(\frac{\theta}{4}\right) - f\left(\frac{\pi}{2} - \frac{\theta}{4}\right)\right]\sin(n\theta)\,d\theta = 0, \quad n \in N.$$

Since the system of functions $\{\sin(n\theta)\}_{n=1}^{\infty}$ forms a basis in the space $L_p(0,\pi)$, $p > 1$, then it follows from the last equality that

$$f(\varphi) = f[(\pi/2) - \varphi], \quad \varphi \in [0, \pi/4]. \tag{45}$$

Now, we consider the first integral from (44) and rewrite it as follows

$$0 = \int_{0}^{\pi/4} f(\varphi) \cos[(4n-3)\varphi]\, d\varphi + \int_{\pi/4}^{\pi/2} f(\varphi) \cos[(4n-3)\varphi]\, d\varphi, \ \ n \in N.$$

Changing the variable of integration by formula $\varphi = (\pi/2) - \theta$ in the second integral, we obtain

$$\int_{0}^{\pi/4} f(\varphi) \cos[(4n-3)\varphi]\, d\varphi + \int_{0}^{\pi/4} f(\pi/2-\theta) \cos[(\pi/2)-(4n-3)\theta]\, d\theta, \ \ n \in N.$$

Hense, introducing the new variable by $\varphi = (\pi/2) - \theta$ in the second integral, we obtain

$$\int_{0}^{\pi/4} f(\varphi) \cos[(4n-3)\varphi]\, d\varphi + \int_{0}^{\pi/4} f(\pi/2-\theta) \cos[(\pi/2)-(4n-3)\theta]\, d\theta, \ \ n \in N.$$

Hence, by virtue of the equality (45), we have

$$\int_{0}^{\pi/4} f(\varphi) \{\cos[(4n-3)\varphi] + \cos[(\pi/2)-(4n-3)\varphi]\}\, d\varphi = 0, \ \ n \in N.$$

Using the cosine sum formula and the reduction formula, we find

$$0 = \int_{0}^{\pi/4} f(\varphi) \cos[(4n-3)\varphi - \pi/4]\, d\varphi = \int_{0}^{\pi/4} f(\varphi) \sin[(4n-3)\varphi + \pi/4]\, d\varphi, \ \ n \in N.$$

Next, making the substitution $4\varphi = \theta$, we obtain

$$\int_{0}^{\pi} f(\theta/4) \sin[(n-3/4)\theta + \pi/4]\, d\theta, \ \ n \in N.$$

It is known that, from the results obtained in the work [34], in particular, the following statements follow.

Corollary 1. *Let* $p \in (1, +\infty)$ *and* $-\frac{1}{p} < \frac{\gamma}{\pi} < 2 - \frac{1}{p}$. *Then the system*

$$\{\sin[(k+\beta/2)\theta + \gamma/2]\}_{k=1}^{\infty}$$

forms a basis in $L_p(0,\pi)$ *(for* $p = 2$ *the Riesz basis) if and only if* $\frac{1}{p} > \frac{\gamma}{\pi} + \beta > \frac{1}{p} - 2$.

According to Corollary 1 , the system of functions $\{\sin[(n-3/4)\theta+\pi/4]\}_{n=1}^{\infty}$ forms a basis in $L_p(0,\pi)$, $p>1$. Therefore, it follows from the last equality that $f(\varphi)\equiv 0,\ 0\le\varphi\le\pi/4$.

Taking this into account and equality (45), we have $f(\varphi)\equiv 0,\ 0\le\varphi\le\pi/2$, whence the assertion of Lemma 1 follows.

Proof of Theorem 1 Let $F(x,y)\in L_2(\Omega_0)$ and

$$\iint\limits_{\Omega_0} F(x,y)\,u_{nm}(x,y)\,dxdy=0,\ n\in N.$$

In this integral we change the variables by formulas $x=r\cos\varphi,\ y=r\sin\varphi$, $r\in[0,1]$,
$\varphi\in[0,\pi/2]$:

$$0=\iint\limits_{\Omega_0} F(x,y)\,u_{nm}(x,y)\,dxdy=$$

$$=c_{nm}\int\limits_0^1\int\limits_0^{\pi/2} f(r,\varphi)\sin[(\pi/2+\varphi)\,\omega_n]\,J_{\omega_n}(\alpha_{nm}r)\,rd\varphi dr=$$

$$=c_{nm}\int\limits_0^1 rJ_{\omega_n}(\alpha_{nm}r)\,dr\int\limits_0^{\pi/2} f(r,\varphi)\{\cos[(4n-3)\varphi]\cup\sin(4n\varphi)\}\,d\varphi,\ n,m\in N,$$

where $f(r,\varphi)=F(r\cos\varphi,r\sin\varphi)$.

It is known that for each fixed $m\in N$ the functions $\sqrt{r}J_{\omega_n}(\alpha_{nm}r)$, $n\in N$ form a complete orthogonal system in the space $L_2(0,1)$. Therefore, from the last equality it follows that

$$\int\limits_0^{\pi/2} f(r,\varphi)\{\cos[(4n-3)\varphi]\cup\sin(4n\varphi)\}\,d\varphi=0,\ n\in N.$$

If we take this into account, then according to Lemma 1, we have $f(r,\varphi)\equiv 0$, i.e. $F(x,y)\equiv 0$ in the space $L_2(\Omega_0)$. Theorem 1 has been proved.

b) Let $q_1=-1$. Then $\omega_n=2n-[(-1)^n+1]/2$.

Similarly to Theorem 1, the following theorem can be proved:

Theorem 2. *The system of eigenfunctions* $\{u_{nm}(x,y)\}_{n,m=1}^{\infty}$ *of problem* $F_\lambda^{(1)}$ *for* $q_1=-1$, *where* $u_{nm}(x,y)$ *is defined by the formula* (41), *is complete in the space* $L_2(\Omega_0)$.

The proof of Theorem 2 is based on the following lemma:

Lemma 2. *The system of functions*

$$\{\sin[(4n-2)\varphi]\}_{n=1}^{\infty} \cup \{\cos[(4n-1)\varphi]\}_{n=1}^{\infty}$$

is complete in the space $L_p(0,\pi/2)$, $p>1$.

Proof. Let $f(\varphi) \in L_p(0,\pi/2)$, $p>1$ and

$$\int_0^{\pi/2} f(\varphi)\sin[(4n-2)\varphi]d\varphi = 0, \quad \int_0^{\pi/2} f(\varphi)\cos[(4n-1)\varphi]d\varphi = 0, \; n \in N. \tag{46}$$

Consider the first integral from (46) and rewrite it in the form

$$0 = \int_0^{\pi/4} f(\varphi)\sin[(4n-2)\varphi]d\varphi + \int_{\pi/4}^{\pi/2} f(\varphi)\sin[(4n-2)\varphi]d\varphi, \; n \in N.$$

In the second integral changing the variable of integration by formula $\varphi = (\pi/2) - \theta$, we have

$$\int_0^{\pi/4} \left[f(\varphi) + f\left(\frac{\pi}{2} - \varphi\right)\right] \sin[(4n-2)\varphi]d\varphi = 0, \; n \in N.$$

Hence, introducing $\varphi = \theta/4$, we obtain

$$\int_0^{\pi} \left[f\left(\frac{\theta}{4}\right) + f\left(\frac{\pi}{2} - \frac{\theta}{4}\right)\right] \sin[(n-1/2)\theta]d\theta = 0, \; n \in N.$$

By virtue of Statement 1, the system of functions $\{\sin[(n-1/2)\theta]\}_{n=1}^{\infty}$ forms a basis in the space $L_p(0,\pi)$, $p>1$. Therefore, it follows from the last equality that

$$f[(\pi/2) - \varphi] = -f(\varphi), \; \varphi \in [0,\pi/4]. \tag{47}$$

Now, let's consider the second integral from (46) and rewrite it as follows

$$0 = \int_0^{\pi/4} f(\varphi)\cos[(4n-1)\varphi]d\varphi + \int_{\pi/4}^{\pi/2} f(\varphi)\cos[(4n-1)\varphi]d\varphi, \; n \in N.$$

Making the substitution $\varphi = (\pi/2) - \theta$ in the second integral, we have

$$\int\limits_0^{\pi/4} f(\varphi)\cos[(4n-1)\varphi]d\varphi + \int\limits_0^{\pi/4} f\left(\frac{\pi}{2}-\theta\right)\cos\left[\frac{\pi}{2}(4n-1)\theta\right]d\theta,\ \ n\in N.$$

Taking into account the equality (47) and the formulas for the difference of cosines, we obtain $\int\limits_0^{\pi/4} f(\varphi)\sin[(4n-1)\theta+\pi/4]d\theta = 0,\ n\in N$. It follows that

$$\int\limits_0^{\pi/4} f\left(\frac{\theta}{4}\right)\sin\left[\left(n-\frac{1}{4}\right)\theta+\frac{\pi}{4}\right]d\theta = 0,\ \ n\in N.$$

According to Corollary 1, the system of functions $\{\sin[(n-(1/4))\theta+(\pi/4)]\}_{n=1}^{\infty}$ forms a basis in the space $L_p(0,\pi),\ p>1$, due to which it follows from the last equality that $f(\theta/4)\equiv 0,\ \theta\in[0,\pi/4]$. Taking this into account and the equality (47), we obtain that $f(\varphi)\equiv 0,\ \varphi\in[0,\pi/2]$, from which the assertion of Lemma 2 follows.

5. Uniqueness of the solution of problem $\mathbf{F}^{(1)}$

First, we prove the uniqueness of the solution to problem $\mathrm{F}^{(1)}$ for $q_1 = 1$. For convenience, we introduce in the domain D_0 cylindrical coordinates (r,φ,z), related to the Cartesian coordinates (x,y,z) by the formulas $x = r\cos\varphi,\ y = r\sin\varphi,\ z = z\ (r=\sqrt{x^2+y^2},\ \varphi = arctg(y/x))$.

In cylindrical coordinates, the domain D_0 goes over to the domain $\tilde{D}_0 = \{(r,\varphi,z):\ r\in(0,1),\ \varphi\in(0,\pi/2),\ z\in(0,c)\}$, and the equations (1) and conditions (2) -(4) are reflected in the following form

$$V_{rr}+\frac{1}{r^2}V_{\varphi\varphi}+\frac{1}{r}V_r+V_{zz}+\frac{2\gamma}{z}V_z = 0,\ \ (r,\varphi,z)\in\tilde{D}_0, \tag{48}$$

$$V\in C(\bar{\tilde{D}})\cap C^1\left(\tilde{D}_0\cup\{r\varphi=0\}\cup\{r=1\}\cup\{\varphi=\pi/2\}\right)\cup C^{2,2,2}_{r,\varphi,z}\left(\tilde{D}_0\right), \tag{49}$$

$$V(1,\varphi,z) = 0,\quad \varphi\in[0,\pi/2],\ \ z\in[0,c], \tag{50}$$

$$V(r,\pi/2,z) = 0,\ \ r\in[0,1],\ \ z\in[0,c], \tag{51}$$

where $V(r,\varphi,z) = U(x,y,z) = U(r\cos\varphi, r\sin\varphi, z)$.

Theorem 3. *If there exists a solution to problem $F^{(1)}$ and the condition $V_\varphi(r,0,z)\sin(\omega_n\pi/2) = -\omega_n V(r,0,z)\cos(\omega_n\pi/2)$ is satisfied, then it is unique.*

Proof. Let $U(x,y,z)\,[V(r,\varphi,z)]$ be a solution of problem $\mathrm{F}^{(1)}$ in the domain $D_0\left[\tilde{D}_0\right]$.

Using the function $V(r,\varphi,z)$ and the eigenfunctions (41), we compose the following function:

$$\vartheta_{nm}(z)=d_{nm}\int\limits_0^1\int\limits_0^{\pi/2} V(r,\varphi,z)\,rJ_{\omega_n}(\alpha_{nm}r)\sin\left[(\pi/2-\varphi)\,\omega_n\right]drd\varphi,\ \ n,m\in N. \tag{52}$$

Based on (52) we introduce the functions

$$\vartheta_{nm}^{\varepsilon_1\varepsilon_2}(z)=d_{nm}\int\limits_{\varepsilon_2}^{1-\varepsilon_2}\int\limits_{\varepsilon_1}^{\pi/2-\varepsilon_1} V(r,\varphi,z)\,rJ_{\omega_n}(\alpha_{nm}r)\sin\left[(\pi/2-\varphi)\,\omega_n\right]drd\varphi, \tag{53}$$

where ε_1 and ε_2 are sufficiently small positive numbers.

It is obvious that $\lim\limits_{\varepsilon_1,\varepsilon_2\to 0}\vartheta_{nm}^{\varepsilon_1\varepsilon_2}(z)=\vartheta_{nm}(z)$.

Here and below, for convenience and compactness, we use the notation $B_q^z\equiv\frac{\partial^2}{\partial z^2}+\frac{2q+1}{z}\frac{\partial}{\partial z}$. Note that the notation B_q^z for $q>-1/2$ coincides with the Bessel operator introduced in [35].

Using the function (53) and the equation (48), we simplify the expression $B_{\gamma-1/2}^z\vartheta_{nm}^{\varepsilon_1\varepsilon_2}(z)$:

$$B_{\gamma-1/2}^z\vartheta_{nm}^{\varepsilon_1\varepsilon_2}(z)=d_{nm}\int\limits_{\varepsilon_2}^{1-\varepsilon_2}\int\limits_{\varepsilon_1}^{\pi/2-\varepsilon_1}\left(B_{\gamma-1/2}^z V\right)rJ_{\omega_n}(\alpha_{nm}r)\sin\left[(\pi/2-\varphi)\omega_n\right]drd\varphi=$$

$$=-d_{nm}\left\{\int\limits_{\varepsilon_1}^{\pi/2-\varepsilon_1}\left[\int\limits_{\varepsilon_2}^{1-\varepsilon_2}V_{rr}rJ_{\omega_n}(\alpha_{nm}r)\,dr\right]\sin\left[(\pi/2-\varphi)\,\omega_n\right]d\varphi+\right.$$

$$+\int\limits_{\varepsilon_1}^{\pi/2-\varepsilon_1}\left[\int\limits_{\varepsilon_2}^{1-\varepsilon_2}V_rJ_{\omega_n}(\alpha_{nm}r)\,dr\right]\sin\left[(\pi/2-\varphi)\,\omega_n\right]d\varphi+$$

$$\left.+\int\limits_{\varepsilon_2}^{1-\varepsilon_2}\left[\int\limits_{\varepsilon_1}^{\pi/2-\varepsilon_1}V_{\varphi\varphi}\sin\left[(\pi/2-\varphi)\,\omega_n\right]d\varphi\right]\frac{1}{r}J_{\omega_n}(\alpha_{nm}r)\,dr\right\}. \tag{54}$$

Applying the rule of integration by parts from (54), we obtain

$$B_{\gamma-1/2}^{z}\vartheta_{nm}^{\varepsilon_1\varepsilon_2}(z)=-d_{nm}\left\{\int\limits_{\varepsilon_1}^{\pi/2-\varepsilon_1}\left\{\left[V_r J_{\omega_n}(\alpha_{nm}r)-V\frac{d}{dr}J_{\omega_n}(\alpha_{nm}r)\right]r\Big|_{r=\varepsilon_2}^{r=1-\varepsilon_2}-\right.\right.$$

$$\left.-\int\limits_{\varepsilon_2}^{1-\varepsilon_2}V(r,\varphi,z)\left(\lambda_{nm}r-\frac{\omega_n^2}{r}\right)J_{\omega_n}(\alpha_{nm}r)\,dr\right\}\sin\left[(\pi/2-\varphi)\,\omega_n\right]d\varphi+$$

$$+\int\limits_{\varepsilon_2}^{1-\varepsilon_2}\left[\left[V_\varphi\sin\left[(\pi/2-\varphi)\,\omega_n\right]+\omega_n V\cos\left[(\pi/2-\varphi)\,\omega_n\right]\right]\Big|_{\varphi=\varepsilon_1}^{\varphi=\pi/2-\varepsilon_1}-\right.$$

$$\left.\left.-\omega_n^2\int\limits_{\varepsilon_1}^{\pi/2-\varepsilon_1}V(r,\varphi,z)\sin\left[(\pi/2-\varphi)\,\omega_n\right]d\varphi\right]\frac{1}{r}J_{\omega_n}(\alpha_{nm}r)\,dr\right\}. \tag{55}$$

Hence, passing to the limit as $\varepsilon_1\to 0$ and $\varepsilon_2\to 0$, taking into account (49), (50), (51) and the equalities $J_{\omega_n}(\alpha_{nm})=0$, as well as the conditions of Theorem 3 and the notation (52), we obtain

$$\vartheta_{nm}''(z)+\frac{2\gamma}{z}\vartheta_{nm}'(z)-\lambda_{nm}\vartheta_{nm}(z)=0,\ \ z\in(0,c)\,.$$

Therefore, the function $\vartheta_{nm}(z)$ satisfies the differential equation (8) for $\lambda=\lambda_{nm}$.

Moreover, due to the boundary conditions (6), it follows from (52), that the function $\vartheta_{nm}(z)$ satisfies the following boundary conditions:

$$\vartheta_{nm}(0)=f_{1nm},\ \ \vartheta_{nm}(c)=f_{2nm}, \tag{56}$$

where

$$f_{lnm}=d_{nm}\int\limits_0^1\int\limits_0^{\pi/2}\tilde{f}_l(r,\varphi)\,rJ_{\omega_n}(\alpha_{nm}r)\sin\left[(\pi/2+\varphi)\,\omega_n\right]dr\,d\varphi,\quad l=\overline{1,2}, \tag{57}$$

$\tilde{f}_l(r,\varphi)=f_l(r\cos\varphi,r\sin\varphi)\,,\ \ l=\overline{1,2}.$

It follows that the function $\vartheta_{nm}(z)$, which has the form (52), is a solution of the equation (8) for $\lambda=\lambda_{nm}$, satisfying the conditions (56). Therefore, subordinating the general solution (42) of the equation (8) (for $\lambda=\lambda_{nm}$) to the conditions (56), we find the coefficients c_7 and c_8 :

$$c_7=\frac{f_{2nm}}{c^{1/2-\gamma}I_{1/2-\gamma}(\alpha_{nm}c)}-\frac{\bar{K}_{1/2-\gamma}(\alpha_{nm}c)\,f_{1nm}}{c^{1/2-\gamma}I_{1/2-\gamma}(\alpha_{nm}c)},\ \ c_8=\frac{2^{1/2+\gamma}(\alpha_{nm})^{1/2-\gamma}f_{1nm}}{\Gamma(1/2-\gamma)},$$

where $\bar{K}_\nu(x) = 2^{1-\nu} x^\nu K_\nu(x) / \Gamma(\nu), \ \nu > 0.$

Substituting these values into (42), we uniquely find $\vartheta_{nm}(z)\,[Z_{nm}(z)]$:

$$\vartheta_{nm}(z) = P_{nm}(z)\, f_{2nm} + \left[\bar{K}_{1/2-\gamma}(\alpha_{nm} z) - P_{nm}(z)\, \bar{K}_{1/2-\gamma}(\alpha_{nm} c)\right] f_{1nm}, \quad (58)$$

where $P_{nm}(z) = (z/c)^{1/2-\gamma}\, I_{1/2-\gamma}(\alpha_{nm} z) \big/ I_{1/2-\gamma}(\alpha_{nm} c).$

To prove Theorem 3, it is sufficient to prove that the homogeneous problem $\mathrm{F}^{(1)}$ has only a trivial solution. Let $f_1(x, y) = f_2(x, y) \equiv 0$, i.e. $\tilde{f}_1(r, \varphi) = \tilde{f}_2(r, \varphi) \equiv 0$. Then $f_{lnm} = 0, \ l = \overline{1, 2}$ for all $n, m \in N$. Based on this, from (58) and (52) it follows that

$$\int\limits_0^1 \int\limits_0^{\pi/2} V(r, \varphi, z)\, r J_{\omega_n}(\alpha_{nm} r) \sin\left[(\pi/2 - \varphi)\, \omega_n\right] dr d\varphi = 0, \ z \in [0, c].$$

Hence, due to the completeness of the system of eigenfunctions $\sqrt{r} J_{\omega_n}(\alpha_{nm} r), \ n \in N$ in the space $L_2(0, 1)$ and $V(r, \varphi, z) \in C(\bar{D}_0)$, it follows

$$\int\limits_0^{\pi/2} V(r, \varphi, z) \sin\left[(\pi/2 - \varphi)\, \omega_n\right] d\varphi = 0, \ r \in [0, 1], \ z \in [0, c].$$

If we take into account the completeness of the system of eigenfunctions $\{\sin\left[(\pi/2 - \varphi)\, \omega_n\right]\}_{n=1}^{\infty}$ in the space $L_2(0, \pi/2)$ and $V(\rho, \varphi, z) = U(x, y, z) \in C\left(\bar{D}_0\right)$, then from the last equality it follows that $V(\rho, \varphi, z) = U(x, y, z) \equiv 0$ in $\bar{D}_0$.

Using this equality and $U(x, y, z) = V(\rho, \varphi, z)$, it is easy to verify that $U(x, +0, z) \equiv 0, \ x \in [0, 1], \ z \in [0, c], \ U_y(x, +0, z) \equiv 0, \ x \in (0, 1), \ z \in (0, c)$.

Then, due to the conditions (2), the equalities are true

$$U(x, -0, z) \equiv 0, \ x \in [0, 1], \ z \in [0, c], \ U_y(x, -0, z) \equiv 0, \ x \in (0, 1), \ z \in (0, c). \quad (59)$$

It follows from Problem 2 that the solution of equation (1) in the domain D_1 satisfying the conditions (59) and (6) (with $f_1(x, y) = f_2(x, y) = 0$) is identically equal to zero, i.e. $U(x, y, z) \equiv 0, \ (x, y, z) \in \bar{\Omega}_1$.

Taking into account $U(+0, y, z) \equiv 0, \ y \in [0, 1], \ z \in [0, c], \ U_x(+0, y, z) \equiv 0, \ y \in (0, 1), \ z \in (0, c)$ and using the conditions (4), (5), (6) (for $f_1(x, y) = f_2(x, y) = 0$), as well as the solution of Problem 1 in the domain D_2, we have $U(x, y, z) \equiv 0, \ (x, y, z) \in \bar{\Omega}_2$. Theorem 3 has been proved.

The uniqueness of the solution to the problem $\mathrm{F}^{(1)}$ for $q_1 = -1$ is proved similarly.

6. Construction and justification of the solution to problem $\mathbf{F}^{(1)}$

Let $q_1 = 1$. Using the function (41) and (58), we construct solutions to the problem $\mathrm{F}^{(1)}$ in the form

$$U(x,y,z) = \begin{cases} \sum\limits_{n=1}^{\infty}\sum\limits_{m=1}^{\infty} U_{nm}^{(0)}(x,y,z), & (x,y,z)\in \bar{D}_0, \ n,m\in N, \\ \sum\limits_{n=1}^{\infty}\sum\limits_{m=1}^{\infty} U_{nm}^{(1)}(x,y,z), & (x,y,z)\in \bar{D}_1, \ n,m\in N, \\ \sum\limits_{n=1}^{\infty}\sum\limits_{m=1}^{\infty} U_{nm}^{(2)}(x,y,z), & (x,y,z)\in \bar{D}_2, \ n,m\in N, \end{cases} \tag{60}$$

where

$$U_{nm}^{(0)}(x,y,z) = R_{nm}(r)\,\Phi_n(\varphi)\,\vartheta_{nm}(z), \tag{61}$$

$$U_{nm}^{(1)}(x,y,z) = \frac{d_{nm}}{2}\left[\left(\frac{x-y}{x+y}\right)^{\frac{\omega_n}{2}} - (-1)^n\left(\frac{x+y}{x-y}\right)^{\frac{\omega_n}{2}}\right] J_{\omega_n}\left(\alpha_{nm}\sqrt{x^2-y^2}\right)\vartheta_{nm}(z), \tag{62}$$

$$U_{nm}^{(2)}(x,y,z) = \frac{d_{nm}}{2}\left[\left(\frac{y-x}{y+x}\right)^{\omega_n/2} - \left(\frac{y+x}{y-x}\right)^{\omega_n/2}\right] J_{\omega_n}\left[\alpha_{nm}\sqrt{y^2-x^2}\right]\vartheta_{nm}(z), \tag{63}$$

$\Phi_n(\varphi)$, $R_{nm}(r)$ and $\vartheta_{nm}(z)$ are defined by the equalities (27), (30) and (58) respectively, and $\bar{J}_\nu(z)$ is the Bessel-Clifford function [36]:

$$\bar{J}_\nu(z) = \Gamma(\nu+1)(z/2)^{-\nu} J_\nu(z) = \sum_{j=0}^{\infty}\frac{\left(-z^2/4\right)^j}{(\nu+1)_j\, j!}. \tag{64}$$

The function $\bar{J}_\nu(z)$ is even and infinitely differentiable. In addition, the equality $\bar{J}_\nu(0) = 1$ and the inequality $\left|\bar{J}_\nu(z)\right| \le 1$ hold for $\nu > -1/2$.

Theorem 4. *Let $\gamma \in (-\infty, 1/2)$ and functions $f_1(x,y) = \tilde{f}_1(r,\varphi)$, $f_2(x,y) = \tilde{f}_2(r,\varphi)$ satisfy the following conditions:*

I. $\tilde{f}_l(r,\varphi) \in C_{r,\varphi}^{4,7}(\bar{\Pi})$, $l=\overline{1,2}$, where $= \{(r,\varphi):\ r\in(0,1),\ \varphi\in(0,\pi/2)\}$;

II. $\left(\partial^{2j}/\partial\varphi^{2j}\right)\tilde{f}_l(r,\pi/2) = 0$, $\left(\partial^k/\partial\varphi^k\right)\tilde{f}_l(r,0) = 0$, $j=\overline{0,3}$, $k=\overline{0,6}$, $l=\overline{1,2}$;

III. $\left(\partial^j/\partial r^j\right)\tilde{f}_l(1,\varphi) = 0$, $\left(\partial^k/\partial r^k\right)\tilde{f}_l(0,\varphi) = 0$, $j=\overline{0,2}$, $k=\overline{0,3}$, $l=\overline{1,2}$.

Then the solution of the problem $F^{(1)}$ exists and is defined by the formula (60).

Before we proceed to the proof of this theorem, we prove some lemmas.

Lemma 3. *For sufficiently large $n \in N$ and $m \in N$, the following estimate holds:*

$$\left|J_{\omega_n+1}\left(\alpha_{nm}\right)\right| \geq c_9 \left(\alpha_{nm}\right)^{-1/2}, \tag{65}$$

where c_9 is a positive constant.

Proof. Since α_{nm} are the zeros of $J_{\omega_n}(r)$, the equality $\int\limits_0^1 r J_{\omega_n}^2\left(\alpha_{nm} r\right) dr = J_{\omega_n+1}^2\left(\alpha_{nm}\right)/2$ is true. It follows from the latter that

$$J_{\omega_n+1}^2\left(\alpha_{nm}\right) = 2\int\limits_0^1 r J_{\omega_n}^2\left(\alpha_{nm} r\right) dr = \frac{2}{\alpha_{nm}^2}\int\limits_0^{\alpha_{nm}} \xi J_{\omega_n}^2(\xi)\, d\xi. \tag{66}$$

By virtue of the asymptotic formula of the Bessel function for sufficiently large values of argument [37]

$$J_\nu(\xi) \approx \left(\frac{2}{\pi\xi}\right)^{1/2} \cos\left(\xi - \frac{\nu\pi}{2} - \frac{\pi}{4}\right), \tag{67}$$

there exists some sufficiently large number $c_0 > 0$ such that for $\xi > c_0$ the equality $\xi J_{1/2-\alpha}^2(\xi) \approx \frac{2}{\pi}\sin^2\left(\xi + \frac{\alpha\pi}{2}\right)$. Then, if we assume that α_{nm} is a sufficiently large number and $\alpha_{nm} > 2\left(c_0+1\right)$, then

$$\int\limits_0^{\alpha_{nm}} \xi J_{\omega_n}^2(\xi)\, d\xi > \int\limits_{c_0}^{\alpha_{nm}} \xi J_{\omega_n}^2(\xi)\, d\xi \approx \frac{2}{\pi}\int\limits_{c_0}^{\alpha_{nm}} \cos^2\left(\xi - \frac{\omega_n\pi}{2} - \frac{\pi}{4}\right) d\xi =$$

$$= \frac{1}{\pi}\alpha_{nm} - \frac{1}{\pi}\left[c_0 + \sin\left(\omega_n\pi - \alpha_{nm} - c_0\right)\sin\left(\alpha_{nm} - c_0\right)\right] \geq \frac{1}{2\pi}\alpha_{nm}.$$

If we take this into account, then from (66), the estimate (65) follows. Lemma 3 has been proved.

Taking into account the estimate (65), we obtain $\left|d_{nm}\right| \leq c_{10}\alpha_{nm}^{1/2}$, where $c_{10} = const > 0$.

Lemma 4. *For the function $R_{nm}(r)$, defined by equalities (30), the following estimates hold for $r \in [0,1]$:*

$$\left|R_{nm}(r)\right| \leq \begin{cases} c_{11}\alpha_{nm}^{1/2}, & \alpha_{nm} r < 1; \\ c_{12}, & \alpha_{nm} r > 1, \end{cases} \tag{68}$$

$$\left|r\frac{d}{dr}\left(r R_{nm}'(r)\right)\right| \leq \begin{cases} c_{11}\omega_n^2\alpha_{nm}^{1/2}, & \alpha_{nm} r < 1; \\ c_{12}\left|\omega_n^2 - \alpha_{nm}^2 r^2\right|, & \alpha_{nm} r > 1. \end{cases} \tag{69}$$

where c_{11}, c_{12} are positive constants.

Proof. It is known that for sufficiently large values of the arguments of the Bessel function of the first kind, the asymptotic formula (67) is valid, and for small values of the arguments [37], we have $J_\nu(\xi) \approx \frac{\xi^\nu}{2^\nu \Gamma(1+\nu)}$.

Taking these formulas into account, we write the function $J_{\omega_n}(\alpha_{nm} r)$ in the form

$$J_{\omega_n}(\alpha_{nm} r) \approx \begin{cases} \frac{(\alpha_{nm} r)^{\omega_n}}{2^{\omega_n}\Gamma(\omega_n+1)}, & \alpha_{nm} r < 1; \\ \left(\frac{2}{\pi \alpha_{nm} r}\right)^{1/2} \cos\left(\alpha_{nm} r - \frac{\omega_n \pi}{2} - \frac{\pi}{4}\right), & \alpha_{nm} r > 1. \end{cases} \tag{70}$$

Taking into account (65) and (70) from (30), we obtain the estimate (68).

It is known that the function $R_{nm}(r)$ satisfies the equation (18) with $\lambda = \lambda_{nm} = \alpha_{nm}^2$ and $\omega = \omega_n$. It follows that $r\frac{d}{dr}(rR'_{nm}(r)) = -\left(\alpha_{nm}^2 r^2 - \omega_n^2\right) R_{nm}(r)$. Then, by estimate (68), estimate (69) holds. Lemma 4 has been proved.

Lemma 5. *The following estimates hold for the functions* (27)

$$|\Phi_n(\varphi)| \le 2/\sqrt{\pi}, \ \left|\Phi'_n(\varphi)\right| \le 2\omega_n/\sqrt{\pi}, \ \left|\Phi''_n(\varphi)\right| \le 2\omega_n^2/\sqrt{\pi}. \tag{71}$$

The validity of the estimate (71) easily follows from the property of trigonometric functions.

Lemma 6. *For any $n, m \in N$ and $\forall z \in [0, c]$, the functions $\vartheta_{nm}(z)$ defined by* (58) *satisfy the estimates*

$$|\vartheta_{nm}(z)| \le 2|f_{1nm}| + |f_{2nm}|, \ \left|B^z_{\gamma-1/2}\vartheta_{nm}(z)\right| \le \alpha_{nm}^2 (2|f_{1nm}| + |f_{2nm}|). \tag{72}$$

Proof. It is known [36] that if $\nu = const > 0$, then

$$\bar{K}_\nu(t) \le 1, \ \bar{K}_\nu(0) = 1. \tag{73}$$

Since $z \in [0, c]$, and $z^{1/2-\gamma} I_{1/2-\gamma}(\alpha_{nm} z)$ is an increasing function [27], then

$$|P_{nm}(z)| = \left|z^{1/2-\gamma} I_{1/2-\gamma}(\alpha_{nm} z) \middle/ \left[c^{1/2-\gamma} I_{1/2-\gamma}(\alpha_{nm} c)\right]\right| \le 1. \tag{74}$$

From the equality (58), according to (73) and (74), the first estimate from (72) follows.

As was shown earlier, the function $\vartheta_{nm}(z)$ satisfies the equation (8) for $\lambda = \lambda_{nm}$. Therefore, the equality $B^z_{\gamma-1/2}\vartheta_{nm}(z) = \lambda_{nm}\vartheta_{nm}(z)$. Hence, by virtue of the first estimate (72), the validity of the second estimate from (72) immediately follows. Lemma 6 is proved.

Lemma 7. *Let the conditions of Theorem 4 be satisfied. Then, for the coefficients* f_{lnm}, $l=\overline{1,2}$, *defined by equality* (57), *the following estimate holds:*

$$|f_{lnm}| \le \begin{cases} c_{13}\omega_n^{-3-\varepsilon_3}\alpha_{nm}^{-3,5-\varepsilon_4}, & \alpha_{nm}r<1; \\ c_{14}\omega_n^{-3-\varepsilon_3}\alpha_{nm}^{-4-\varepsilon_4}, & \alpha_{nm}r>1. \end{cases} \tag{75}$$

where c_{13}, c_{14} *are some positive constants, and* ε_3 *and* ε_4 *are sufficiently small positive numbers.*

Proof. The coefficient f_{lnm}, $l=\overline{1,2}$ is represented in the form

$$f_{lnm} = d_{nm}\int\limits_0^1 F_{ln}(r)\, rJ_{\omega_n}(\alpha_{nm}r)\,dr, \tag{76}$$

where $F_{ln}(r) = \int\limits_0^{\pi/2} \tilde{f}_l(r,\varphi)\sin\left[(\pi/2-\varphi)\,\omega_n\right]d\varphi$.

First, consider the function $F_{ln}(r)$ and rewrite it as

$$F_{ln}(r) = \frac{1}{\omega_n}\int\limits_0^{\pi/2}\tilde{f}_l(r,\varphi)\frac{d}{d\varphi}\left\{\cos\left[(\pi/2-\varphi)\,\omega_n\right]\right\}d\varphi.$$

Applying the rule of integration by parts four times, from the last, we obtain

$$F_{ln}(r) = \frac{1}{\omega_n}\tilde{f}_l(r,\varphi)\cos\left[(\pi/2-\varphi)\,\omega_n\right]\Big|_{\varphi=0}^{\varphi=\pi/2} +$$

$$+\frac{1}{\omega_n^2}\frac{\partial}{\partial\varphi}\tilde{f}_l(r,\varphi)\sin\left[\left(\frac{\pi}{2}-\varphi\right)\omega_n\right]\Big|_{\varphi=0}^{\varphi=\frac{\pi}{2}} - \frac{1}{\omega_n^3}\frac{\partial^2}{\partial\varphi^2}\tilde{f}_l(r,\varphi)\cos\left[\left(\frac{\pi}{2}-\varphi\right)\omega_n\right]\Big|_{\varphi=0}^{\varphi=\frac{\pi}{2}} +$$

$$-\frac{1}{\omega_n^4}\frac{\partial^3}{\partial\varphi^3}\tilde{f}_l(r,\varphi)\sin\left[\left(\frac{\pi}{2}-\varphi\right)\omega_n\right]\Big|_{\varphi=0}^{\varphi=\frac{\pi}{2}} + \frac{1}{\omega_n^5}\frac{\partial^4}{\partial\varphi^4}\tilde{f}_l(r,\varphi)\cos\left[\left(\frac{\pi}{2}-\varphi\right)\omega_n\right]\Big|_{\varphi=0}^{\varphi=\frac{\pi}{2}} +$$

$$+\frac{1}{\omega_n^6}\frac{\partial^5}{\partial\varphi^5}\tilde{f}_l(r,\varphi)\sin\left[\left(\frac{\pi}{2}-\varphi\right)\omega_n\right]\Big|_{\varphi=0}^{\varphi=\frac{\pi}{2}} - \frac{1}{\omega_n^7}\frac{\partial^6}{\partial\varphi^6}\tilde{f}_l(r,\varphi)\cos\left[\left(\frac{\pi}{2}-\varphi\right)\omega_n\right]\Big|_{\varphi=0}^{\varphi=\frac{\pi}{2}} +$$

$$+\frac{1}{\omega_n^7}\int\limits_0^{\pi/2}\frac{\partial^7}{\partial\varphi^7}\tilde{f}_l(r,\varphi)\cos\left[(\pi/2-\varphi)\,\omega_n\right]d\varphi.$$

By the conditions of Theorem 4, the terms outside the integral in the last equality are equal to zero. Therefore,

$$F_{ln}(r) = \frac{1}{\omega_n^7} \int\limits_0^{\pi/2} \frac{\partial^7}{\partial\varphi^7} \tilde{f}_l(r,\varphi) \cos\left[(\pi/2-\varphi)\,\omega_n\right] d\varphi. \tag{77}$$

From here, based on the first part of Theorem 4, we conclude that the integral in (77) exists and $F_{ln}(r) \in C[0,1]$.

Since the function $\left(\partial^7/\partial\varphi^7\right)\tilde{f}_l(r,\varphi)$ with respect to the variable φ is continuous on $[0,\pi/2]$, then, opplying Riemann's lemma [39] to (77), we have

$$\lim_{n\to\infty} \int\limits_0^{\pi/2} \frac{\partial^7}{\partial\varphi^7} \tilde{f}_l(r,\varphi) \cos\left[(\pi/2-\varphi)\,\omega_n\right] d\varphi = 0. \tag{78}$$

Now, let us consider the coefficient f_{lnm}, defined by the equality (76).

Using the equality $rJ_{\omega_n}(\alpha_{nm}r) = \frac{r^{-\omega_n}}{\alpha_{nm}}\frac{d}{dr}\left[r^{\omega_n+1}J_{\omega_n+1}(\alpha_{nm}r)\right]$, we write the coefficient f_{lnm} as

$$f_{lnm} = \frac{d_{nm}}{\alpha_{nm}} \int\limits_0^1 F_{ln}(r)\, r^{-\omega_n} d\left[r^{\omega_n+1}J_{\omega_n+1}(\alpha_{nm}r)\right].$$

Applying the rule of integration by parts, from the latter we obtain

$$f_{lnm} = \frac{d_{nm}}{\alpha_{nm}} F_{ln}(r)\, rJ_{\omega_n+1}(\alpha_{nm}r)\Big|_{r=0}^{r=1} -$$

$$-\frac{d_{nm}}{\alpha_{nm}} \int\limits_0^1 r^{\omega_n+1} J_{\omega_n+1}(\alpha_{nm}r) \frac{d}{dr}\left[r^{-\omega_n}F_{ln}(r)\right] dr. \tag{79}$$

By virtue of equality

$$r^{\omega_n+1}J_{\omega_n+1}(\alpha_{nm}r) = -\frac{r^{2\omega_n+1}}{\alpha_{nm}}\frac{d}{dr}\left[r^{-\omega_n}J_{\omega_n}(\alpha_{nm}r)\right],$$

we rewrite (79) in the form

$$f_{lnm} = \frac{d_{nm}}{\alpha_{nm}} F_{ln}(r)\, rJ_{\omega_n+1}(\alpha_{nm}r)\Big|_{r=0}^{r=1} +$$

$$+\frac{d_{nm}}{\alpha_{nm}^2} \int\limits_0^1 r^{2\omega_n+1} \frac{d}{dr}\left[r^{-\omega_n}F_{ln}(r)\right] d\left[r^{-\omega_n}J_{\omega_n}(\alpha_{nm}r)\right].$$

Hence, applying the rule of integration by parts once again, we have

$$f_{lnm} = \frac{d_{nm}}{\alpha_{nm}} \ F_{ln}\left(r\right) r J_{\omega_n+1}\left(\alpha_{nm} r\right)\Big|_{r=0}^{r=1} +$$

$$+\frac{d_{nm}}{\alpha_{nm}^2} \ r^{\omega_n+1} J_{\omega_n}\left(\alpha_{nm} r\right) \frac{d}{dr}\left[r^{-\omega_n} F_{ln}\left(r\right)\right]\Bigg|_{r=0}^{r=1} -$$

$$-\frac{d_{nm}}{\alpha_{nm}^2} \int\limits_0^1 r^{-\omega_n} J_{\omega_n}\left(\alpha_{nm} r\right) \frac{d}{dr} r^{2\omega_n+1} \frac{d}{dr}\left[r^{-\omega_n} F_{ln}\left(r\right)\right] dr.$$

Continuing this process, i.e. applying the rule of integration by parts two more times, we have

$$f_{lnm} = \frac{d_{nm}}{\alpha_{nm}} \ r J_{\omega_n+1}\left(\alpha_{nm} r\right) F_{ln}\left(r\right)\Big|_{r=0}^{r=1} +$$

$$+\frac{d_{nm}}{\alpha_{nm}^2} \ r^{\omega_n+1} J_{\omega_n}\left(\alpha_{nm} r\right) \frac{d}{dr}\left[r^{-\omega_n} F_{ln}\left(r\right)\right]\Bigg|_{r=0}^{r=1} -$$

$$-\frac{d_{nm}}{\alpha_{nm}^3} \ r^{-\omega_n} J_{\omega_n+1}\left(\alpha_{nm} r\right) \frac{d}{dr} r^{2\omega_n+1} \frac{d}{dr}\left[r^{-\omega_n} F_{ln}\left(r\right)\right]\Bigg|_{r=0}^{r=1} -$$

$$-\frac{d_{nm}}{\alpha_{nm}^4} \ r^{\omega_n+1} J_{\omega_n}\left(\alpha_{nm} r\right) \frac{d}{dr} r^{-2\omega_n-1} \frac{d}{dr} r^{2\omega_n+1} \frac{d}{dr}\left[r^{-\omega_n} F_{ln}\left(r\right)\right]\Bigg|_{r=0}^{r=1} +$$

$$+\frac{d_{nm}}{\alpha_{nm}^4} \int\limits_0^1 r^{-\omega_n} J_{\omega_n}\left(\alpha_{nm} r\right) \frac{d}{dr} r^{2\omega_n+1} \frac{d}{dr} r^{-2\omega_n-1} \frac{d}{dr} r^{2\omega_n+1} \frac{d}{dr}\left[r^{-\omega_n} F_{ln}\left(r\right)\right] dr. \tag{80}$$

By virtue of $J_{\omega_n}\left(\alpha_{nm}\right) = 0$ and the conditions of Theorem 4, the terms outside the integral in (80) are equal to zero. Hence,

$$f_{lnm} = \frac{d_{nm}}{\alpha_{nm}^4} \int\limits_0^1 r^{-\omega_n} J_{\omega_n}\left(\alpha_{nm} r\right) \frac{d}{dr} r^{2\omega_n+1} \frac{d}{dr} r^{-2\omega_n-1} \frac{d}{dr} r^{2\omega_n+1} \frac{d}{dr}\left[r^{-\omega_n} F_{ln}\left(r\right)\right] dr.$$

Using function expansion

$$r^{-\omega_n} \frac{d}{dr} r^{2\omega_n+1} \frac{d}{dr} r^{-2\omega_n-1} \frac{d}{dr} r^{2\omega_n+1} \frac{d}{dr}\left[r^{-\omega_n} F_{ln}\left(r\right)\right],$$

it's easy to see that

$$f_{lnm}=\frac{d_{nm}}{\alpha_{nm}^{4}}\int\limits_{0}^{1} rJ_{\omega_n}\left(\alpha_{nm}r\right)\left[\omega_n^2\left(\omega_n^2-4\right)r^{-4}F_{ln}\left(r\right)+\left(2\omega_n^2+1\right)r^{-3}F_{ln}'\left(r\right)-\right.$$

$$\left.-\left(2\omega_n^2+1\right)r^{-2}F_{ln}''\left(r\right)+2r^{-1}F_{ln}'''\left(r\right)+F_{ln}^{(4)}\left(r\right)\right]dr. \tag{81}$$

From here, based on the conditions of heorem 4, it follows that the expressions in square brackets belong to the class $C\left(\bar{\Pi}\right)$. Taking this into account and $J_{\omega_n}\left(\alpha_{nm}r\right)\in C\left[0,1\right]$, we conclude that the integral in (81) exists. In addition, we have obtained an analogue of the property of integrals containing trigonometric functions (see formula (78)) [39], i.e.

$$\lim_{m\to\infty}\int\limits_{0}^{1} rJ_{\omega_n}\left(\alpha_{nm}r\right)\left[\omega_n^2\left(\omega_n^2-4\right)r^{-4}F_{ln}\left(r\right)+\left(2\omega_n^2+1\right)r^{-3}F_{ln}'\left(r\right)-\right.$$

$$\left.-\left(2\omega_n^2+1\right)r^{-2}F_{ln}''\left(r\right)+2r^{-1}F_{ln}'''\left(r\right)+F_{ln}^{(4)}\left(r\right)\right]dr=0. \tag{82}$$

Hence, taking into account (77) and the form of the coefficient d_{nm}, we have

$$f_{lnm}=\frac{-1}{\omega_n^8\alpha_{nm}^4}\int\limits_0^1\int\limits_0^{\pi/2}R_{nm}\left(r\right)\Phi_n'\left(\varphi\right)\left[\omega_n^2\left(\omega_n^2-4\right)r^{-3}+\left(2\omega_n^2+1\right)r^{-2}\frac{\partial}{\partial r}-\right.$$

$$\left.-\left(2\omega_n^2+1\right)r^{-1}\frac{\partial^2}{\partial r^2}+2\frac{\partial^3}{\partial r^3}+r\frac{\partial^4}{\partial r^4}\right]\frac{\partial^8}{\partial\varphi^8}\tilde{f}_l\left(r,\varphi\right)d\varphi dr. \tag{83}$$

By the hypothesis of Theorem 4, we conclude that the integrand in (83) is continuous in $\bar{\Pi}$, and the iterated integral in (83) exists.

Taking into account the estimates (68), (71), (78) and (82) from (83), we obtain the estimates (75). Lemma 7 has been proved.

Based on (75), the estimate (72) can be rewritten as follows

$$\left|\vartheta_{nm}\left(z\right)\right|\le\left\{\begin{array}{ll} c_{15}\omega_n^{-3-\varepsilon_3}\alpha_{nm}^{-3.5-\varepsilon_4}, & \alpha_{nm}r<1;\\ c_{16}\omega_n^{-3-\varepsilon_3}\alpha_{nm}^{-4-\varepsilon_4}, & \alpha_{nm}r>1,\end{array}\right. \tag{84}$$

$$\left|B_{\gamma-1/2}^{z}\vartheta_{nm}\left(z\right)\right|\le\left\{\begin{array}{ll} c_{15}\omega_n^{-3-\varepsilon_3}\alpha_{nm}^{-1.5-\varepsilon_4}, & \alpha_{nm}r<1;\\ c_{16}\omega_n^{-3-\varepsilon_3}\alpha_{nm}^{-2-\varepsilon_4}, & \alpha_{nm}r>1.\end{array}\right. \tag{85}$$

where c_{15} and c_{16} are positive constants.

According to [27], for sufficiently large m, for the mth positive root of the equations $J_{\omega_n}(x)=0$, the relation $\alpha_{nm}\approx\pi\left(m+\omega_n/2-1/4\right)$ holds.

For sufficiently large n, we have $\omega_n\approx 2n$. Then, for sufficiently large n and m, α_{nm} is equivalent to $\pi\left(m+n\right).$

Proof of Theorem 4. To prove the theorem, it is sufficient to prove the uniform convergence of the series (60) in $\bar{D}$, and also of the series $r\left[\frac{\partial}{\partial r}rV_r\left(r,\varphi,z\right)\right]$, $V_{\varphi\varphi}$, $B^z_{\gamma-1/2}U$, U_{xx}, U_{yy}, in any compact $K\subset D_0\cup D_1\cup D_2$.

According to the estimates (68), (71), (84) and the relations $\omega_n\approx 2n,\ \alpha_{nm}\approx\pi\left(m+n\right)>\pi m$, the series (60) in the domain $\bar{\tilde{D}}_0$ are estimated by the following products of numerical series

$$\begin{cases} c_{17}\sum\limits_{n=1}^{\infty}n^{-3-\varepsilon_3}\sum\limits_{m=1}^{\infty}m^{-3-\varepsilon_4} & \alpha_{nm}r<1,\\ c_{18}\sum\limits_{n=1}^{\infty}n^{-3-\varepsilon_3}\sum\limits_{m=1}^{\infty}m^{-4-\varepsilon_4} & \alpha_{nm}r>1,\end{cases} \tag{86}$$

where c_{17} and c_{18} are positive constants.

According to the estimates (68), (69), (71), (84), (85) and the relations $\omega_n\approx 2n,\ \alpha_{nm}\approx\pi\left(m+n\right)>\pi m$, the series $r\left[\frac{\partial}{\partial r}rV_r\left(r,\varphi,z\right)\right]$, $V_{\varphi\varphi}$ and $B^z_{\gamma-1/2}V$ in the region $D_0\left[\tilde{D}_0\right]$ are estimated by the following products of the numerical series, respectively

$$\begin{cases} c_{19}\sum\limits_{n=1}^{\infty}n^{-1-\varepsilon_3}\sum\limits_{m=1}^{\infty}m^{-3-\varepsilon_4} & \alpha_{nm}r<1,\\ c_{20}\left[\sum\limits_{n=1}^{\infty}n^{-1-\varepsilon_3}\sum\limits_{m=1}^{\infty}m^{-4-\varepsilon_4}+\sum\limits_{n=1}^{\infty}n^{-3-\varepsilon_3}\sum\limits_{m=1}^{\infty}m^{-2-\varepsilon_4}\right] & \alpha_{nm}r>1,\end{cases} \tag{87}$$

$$\begin{cases} c_{21}\sum\limits_{n=1}^{\infty}n^{-1-\varepsilon_3}\sum\limits_{m=1}^{\infty}m^{-3-\varepsilon_4} & \alpha_{nm}r<1,\\ c_{22}\sum\limits_{n=1}^{\infty}n^{-1-\varepsilon_3}\sum\limits_{m=1}^{\infty}m^{-4-\varepsilon_4} & \alpha_{nm}r>1,\end{cases} \tag{88}$$

$$\begin{cases} c_{23}\sum\limits_{n=1}^{\infty}n^{-3-\varepsilon_3}\sum\limits_{m=1}^{\infty}m^{-1-\varepsilon_4} & \alpha_{nm}r<1,\\ c_{24}\sum\limits_{n=1}^{\infty}n^{-3-\varepsilon_3}\sum\limits_{m=1}^{\infty}m^{-2-\varepsilon_4} & \alpha_{nm}r>1,\end{cases} \tag{89}$$

where c_j, $j=\overline{19,24}$ are some positive constants.

Since both factors of the numerical series in (86)-(89) converge, the series (60) converge absolutely and uniformly in $\bar{\tilde{D}}_0$, and the series $r\left[\frac{\partial}{\partial r}rV_r\left(r,\varphi,z\right)\right]$, $V_{\varphi\varphi}$ and $B^z_{\gamma-1/2}V$ converge on each compact $K\subset\tilde{D}_0$.

Now consider the series (60) in the domain $\bar{D}_1$.

Let $\alpha_{nm}\xi < 1$. As ξ we can take any function from the interval $(0,1)$. Then, by (70), the function (62) can be written as

$$U_{nm}^{(1)}(x,y,z) = \frac{d_{nm}\alpha_{nm}^{\omega_n}}{2^{\omega_n+1}\Gamma(\omega_n+1)}\left[(x-y)^{\omega_n} - (-1)^n (x+y)^{\omega_n}\right]\vartheta_{nm}(z). \quad (90)$$

In the region $\bar{\Omega}_1$, the following estimates hold:

$$0 \le x-y \le 1, \;\; 0 \le x+y \le 1,$$

$$\left|(x-y)^{\omega_n} - (-1)^n (x+y)^{\omega_n}\right| \le \left|(x-y)^{\omega_n} + (x+y)^{\omega_n}\right| \le |x-y+x+y|^{\omega_n} \le 2^{\omega_n} x^{\omega_n}.$$

By virtue of these inequalities, as well as the estimate $|d_{nm}| \le c_{10}\alpha_{nm}^{1/2}$ and (84), we estimate the function (90):

$$\left|U_{nm}^{(1)}(x,y,z)\right| \le c_{25}\alpha_{nm}^{1/2}(\alpha_{nm}x)^{\omega_n}\omega_n^{-3-\varepsilon_3}\alpha_{nm}^{-3,5-\varepsilon_4} \le c_{25}\omega_n^{-3-\varepsilon_3}\alpha_{nm}^{-3-\varepsilon_4}, \quad (91)$$

where c_{25} is a positive constant.

By the estimate (91) and the relations $\omega_n \approx 2n,\; \alpha_{nm} \approx \pi(m+n) > \pi m$, the series (60) in the domain D_1 is uniformly convergent by the Weierstrass M-test [38].

Let $\alpha_{nm}\xi > 1$. Then by the estimate (70) and the relations $\omega_n \approx 2n,\; \alpha_{nm} \approx \pi(m+n) > \pi m$, and also the estimate $|d_{nm}| \le c_{10}\alpha_{nm}^{1/2}$ and (84), the function (62) is estimated as $\left|U_{nm}^{(1)}(x,y,z)\right| \le n^{-3-\varepsilon_3}m^{-4-\varepsilon_4}$.

Here, also according to the Weierstrass M-test, the series (60) in the domain D_1 is uniformly convergent.

Similarly, we prove the uniform convergence of series (60) in the domain D_2.

Here, for $\alpha_{nm}\xi < 1$ $(0 \le \xi \le 1)$, we have that

$$U_{nm}^{(2)}(x,y,z) = \frac{d_{nm}}{2}\frac{\alpha_{nm}^{\omega_n}}{2^{\omega_n}\Gamma(\omega_n+1)}\left[(-y+x)^{\omega_n} - (-y-x)^{\omega_n}\right]\vartheta_{nm}(z).$$

Due to these inequalities, as well as the estimate $|d_{nm}| \le c_{10}\alpha_{nm}^{1/2}$ and (84), we estimate the function (90):

$$\left|U_{nm}^{(2)}(x,y,z)\right| \le c_{26}\alpha_{nm}^{1/2}(-\alpha_{nm}y)^{\omega_n}\omega_n^{-3-\varepsilon_3}\alpha_{nm}^{-3.5-\varepsilon_4} \le c_{26}\omega_n^{-3-\varepsilon_3}\alpha_{nm}^{-3-\varepsilon_4}. \quad (92)$$

where $c_{26} is$ a positive constant.

By virtue of the estimate (92) and the relations $\omega_n \approx 2n,\; \alpha_{nm} \approx \pi(m+n) > \pi m$, the series (60) in the domain D_2 is uniformly convergent by the Weierstrass M-test.

The estimate for the function (63) is proved similarly for $\alpha_{nm}\xi > 1$.

A similar method is used to prove that the functions U_{xx} and U_{yy} in the compact $K \subset D_1 \cup D_2$ are estimated in absolute value by the following product of numerical series

$$c_{27} \sum_{n=1}^{\infty} n^{-1-\varepsilon_3} \sum_{m=1}^{\infty} m^{-3-\varepsilon_4}, \tag{93}$$

where c_{27} is a positive constant.

Both factors of the numerical series in (93) converge, then the series U_{xx} and U_{yy} converge absolutely and uniformly on each compact $K \subset D_1 \cup D_2$. Therefore, the function $U(x, y, z)$, defined by the series (60), satisfies all the conditions of Problem $\mathrm{F}^{(1)}$. Theorem 4 is proved.

The existence of a solution to the problem posed for $q_1 = -1$ is proved similarly. This completes the study of Problem $\mathrm{F}^{(1)}$.

References

[1] Bitsadze, A. V. (1956). On a problem of Frankl. Doklady Akaemii Nauk SSSR, 109(6), 1091-1094. (in Russian)

[2] Bitsadze, A. V. (1956). On the uniqueness of the solution of the Frankl problem for the Chaplygin equation. Doklady Akaemii Nauk SSSR, 112(3), 375-376. (in Russian)

[3] Devingtal, Yu. V. (1958). On the existence of the solution of a problem of F. I. Frankl. Doklady Akademii Nauk SSSR, 119(1), 15-18. (in Russian)

[4] Devingtal, Yu. V. (1958). On the existence and uniqueness of the solution of a problem of F. I. Frankl. Izvestiya Vysshikh Uchebnykh Zavedeniy. Matematika, 2(3), 39-51. (in Russian)

[5] Devingtal, Yu. V. (1959). Concerning the existence and uniqueness of the solution of a problem of Frankl. Uspekhi Matemeticheskikh Nauk, 14(1), 177-182. (in Russian)

[6] Moiseev, E. I. (1992). On the solution of the Frankl problem in a special domain. Differential Equations, 28(4), 721-723. (in Russian)

[7] Moiseev, E. I., & Abbasi, N. (2008). On the completeness of eigenfunctions of the Frankl problem with the oddness condition. Differential Equations, 44, 408-412. https://doi.org/10.1134/S0012266108030117

[8] Moiseev, E. I., & Abbasi, N. (2008). On the basis property of eigenfunctions of the Frankl problem with a nonlocal parity condition. Differential Equations, 44, 855-860. https://doi.org/10.1134/S0012266108060098

[9] Moiseev, E. I., & Abbasi, N. (2009). Basis property of the eigenfunctions of a generalized gasdynamic Frankl problem with a nonlocal evenness condition and with a discontinuity in the solution gradient. Differential Equations, 45, 1485-1490. https://doi.org/10.1134/S0012266109100115

[10] Moiseev, E. I., & Ambartsumyan, V. E. (2009). On the basis property of eigenfunctions of the Frankl problem with a nonlocal parity condition of the second kind. Differential Equations, 45, 1769-1774. https://doi.org/10.1134/S0012266109120076

[11] Moiseev, E. I., & Ambartsumyan, V. E. (2010). On the basis property of the eigenfunctions of the Frankl problem with nonlocal evenness and oddness conditions of the second kind. Doklady Mathematics, 81, 429–433. https://doi.org/10.1134/S1064562410030257

[12] Burmistrov, B. N. (1962). On a problem of F. I. Frankl for a system of equations. Proc. Second Scientific Conference of Mathematical Departments of Pedagogical Institues in the Volga Region, 1, 14-19. (in Russian)

[13] Zhegalov, V. I. (1979). The Frankl problem with a shift. Soviet Mathematics, 23(9), 10-19. (in Russian)

[14] Klokov, V. V. (1966). On the uniqueness of the solution of the generalized Frankl problem. Proceedings of Seminar on Boundary-Value Problems, 3, 66-67. (in Russian)

[15] Mayorov, I. V. (1966). On the extremum principle for a problem of Frankl. Sibirskiy Matematicheskiy Zhurnal, 7(5), 1068-1075. (in Russian)

[16] Allakhyarova, N. E. (2023). Eigenvalues of Fredholm type limit integral equations in the space of Bohr almost periodic functions. Journal of Contemporary Applied Mathematics, 13(1), 71–82.

[17] Hadiyeva, S. S. (2024). Basis property of the system of eigenfunctions corresponding to a problem with a spectral parameter in the boundary condition. Journal of Contemporary Applied Mathematics, 14(2), 8-11. https://doi.org/10.69624/2222-5498.2024.2.8

[18] Salakhitdinov, M. S., & Islomov, B. (1990). On a three-dimensional analogue of the Tricomi problem for an equation of mixed type. Doklady Academii Nauk SSSR, 311(4), 797-801. (in Russian)

[19] Moiseev, E. I., & Nefedov, P. V. (2013). Frankl problem for the Lavrent'ev-Bitsadze equation in a 3D-domain. Integral Transforms and Special Functions, 24(7), 554-560.

[20] Urinov, A. K., & Karimov, K. T. (2020). The Dirichlet problem for an elliptic equation with singular coefficients in a semi-cylindrical domain. Lobachevskii Journal of Mathematics, 41(9), 1891-1902. https://doi.org/10.1134/S1995080220090292

[21] Yuldashev, T. K. (2020). Nonlocal inverse problem for a pseudohyperbolic-pseudoelliptic type integro-differential equations. Axioms, 9(2), 1-21. https://doi.org/10.3390/axioms9020045

[22] Yuldashev, T. K. (2020). On a boundary-value problem for a fourth-order partial integro-differential equation with degenerate kernel. Journal of Mathematical Science, 245(4), 508-523. https://doi.org/10.1007/s10958-020-04707-2

[23] Yuldashev, T. K. (2019). Determination of the coefficient in nonlocal problem for a Boussinesq type integro-differential equation with degenerate kernel. Vladikavkazskii Matematicheskii Zhurnal, 21(2), 67-84. (in Russian) https://doi.org/10.23671/VNC.2019.2.32118

[24] Yuldashev, T. K., & Kadirkulov, B. J. (2020). Boundary value problem for weak nonlinear partial differential equations of mixed type with fractional Hilfer operator. Axioms, 9(2), 1-19. https://doi.org/10.3390/axioms9020068

[25] Yuldashev, T. K., Mamedov, Kh. R., & Babayev, M. M. (2024). Dependence on the parameters of the solution of a mixed problem for a nonlinear integro-differential equation of the fifth order with a degenerate kernel. Journal of Contemporary Applied Mathematics, 14(2), 38-60. https://doi.org/10.69624/2222-5498.2024.2.38

[26] Smirnov, M. M. (1970). Equations of mixed type. Moscow, Nauka, 1970. 798 p. (in Russian)

[27] Watson, G. N. (1944). A Treatise on the theory of Bessel functions. Cambridge, Cambridge Univ. Press, 1944. 799 p.

[28] Frankl, F. I. (1956). Subsonic flow past a profile with supersonic zone terminated with a normal compression shock. Prikladnaya Matematika i Mekhanika, 20(2), 196-202. (in Russian)

[29] Karimov, K. T., & Shokirov, A. M. (2024). A problem for a three-dimensional equation of mixed type with singular coefficient. International Journal of Applied Mathematics, 37(1), 73-92. http://dx.doi.org/10.12732/ijam.v37i1.7

[30] Urinov, A. K., & Karimov, K. T. (2016). Trikomi problem for a three-dimensional equation of mixed type with three singular coefficients. Bulletin of National University of Uzbekistan, 2, 14-25.

[31] Sabitov, K. B. Tikhomirov, V. V. (1990). On the construction of eigenvalues and functions of one Frankl gas-dynamic problem. Math. modeling, 2(10), 100-109. (in Russian)

[32] Bateman, H., & Erdeyi, A. (1953). Hiper transcendental functions. New York, McGraw-Hill, 1953, 325 p.

[33] Karimov, K. T. (2024). On a problem for a three-dimensional elliptic equation with bessel operators. Chelyabinsk Phys. and Math. J., 9(3), 375-388. http://doi.org/10.47475/2500-0101-2024-9-3-375-388

[34] Moiseev, E. I. (1987). On the basis property of one system of sines. Diff. equations, 23(1), 177-179. (in Russian)

[35] Kipriyanov, I. A. (1997). Singular elliptic boundary-value problems. Moscow, Nauka, 1997, 204 p. (in Russian).

[36] Kapilevich, M. B. (1952). On one equation of mixed elliptic-hyperbolic type. Matematicheskiy Zbornik, (1), 11-38. (in Russian)

[37] Lebedev, N. N. (1963). Special functions and their applications. Moscow, Fizmatlit, 1963, 358 p. (in Russian).

[38] Fichtenholz, G. M. (2001). Course of differential and integral calculus. Moscow, Fizmatlit, 2001, V. 2. 800 p. (in Russian)

[39] Tolstov, G. P. (1980). Fourier series. Moscow, Nauka, 1980, 384 p. (in Russian)

Kamoliddin T. Karimov
1. Fergana State University, Murabbiylar Street 19, Fergana, Uzbekistan;
2. V.I.Romanovskiy Institute of Mathematics, Uzbekistan Academy of Sciences, Tashkent, Uzbekistan;
E-mail: karimovk80@mail.ru

Asrorjon M. Shokirov
Fergana State University, Murabbiylar Street 19, Fergana, Uzbekistan;
E-mail: asrorshokirov87@gmial.com

Received: 11 March 2025
Accepted: 08 August 2025