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Sesquilinear A-form and Bilinear A-form

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Abstract. In this paper, we introduce and investigate the concepts of *sesquilinear A-forms* and *bilinear A-forms*, which extend the classical notions of sesquilinear and bilinear forms to modules over C^*-algebras. Rigorous definitions for both left and right sesquilinear A-forms, as well as bilinear A-forms, are provided together with illustrative examples. Several structural properties of these A-forms are established, including their relationships with corresponding quadratic A-forms and inequalities analogous to the Cauchy–Schwarz inequality. The results presented here form a foundational framework for further research in A-valued functional analysis and operator theory.

Key words and phrases: Sesquilinear form, sesquilinear A-form, bilinear A-form, C^*-algebra, A-module.
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1. Introduction

In mathematics, a **sesquilinear form** [11, 2, 3, 8, 12, 6] is a generalization of the bilinear form, which in turn generalizes the **inner product** in Euclidean space. Sesquilinear forms play a crucial role in the theory of linear operators on Hilbert spaces and appear in various areas of analysis. For example, sesquilinear forms are used in finite measures on σ-algebras, positive definite kernels on Banach spaces, and positive linear functionals on (Banach)*-algebras [13, 5].

Sesquilinear forms play a central role in functional analysis and operator theory, as they generalize the classical notion of bilinear forms and provide the analytical foundation for studying operators on Hilbert spaces. A bilinear form is linear in each argument, while a sesquilinear form is linear in one argument and conjugate-linear in the other. The term *sesquilinear* originates from the Latin prefix *sesqui-* meaning "one and a half," indicating the asymmetric linearity in its arguments.

A fundamental example of a sesquilinear form on a complex vector space V is a mapping

$$S : V \times V \longrightarrow \mathbb{C},$$

which is linear in one argument and conjugate-linear in the other. Sesquilinear forms appear in various contexts of mathematical physics, spectral theory, and geometry. They are used, for instance, in defining positive definite kernels, in describing finite measures on σ-algebras, and in the study of linear functionals on Banach $*$-algebras.

In the present work, we extend this notion from vector spaces over $\mathbb{C}$ to A-modules over a C^*-algebra A. We define and analyze *left* and *right* sesquilinear A-forms, as well as bilinear A-forms, and derive several identities and inequalities analogous to those in the classical theory. These generalizations allow the study of forms taking values in a noncommutative algebra rather than in the field of complex numbers, thereby broadening the scope of sesquilinear form theory to include operator-valued settings.

2. Sesquilinear Form on a Vector Space

Definition 1. [11], [4] *Let H be a vector space over $\mathbb{C}$. A mapping $S : H \times H \to \mathbb{C}$ is called a sesquilinear form on H if for all $f, g, h \in H$ and $a, b \in \mathbb{C}$, the following conditions hold*

(1) $S(af + bg, h) = aS(f, h) + bS(g, h)$

(2) $S(f, ag + bh) = \bar{a}S(f, g) + \bar{b}S(f, h)$*,*

If condition (1) *holds without complex conjugation, then S is called a* **bilinear form** *on H. In particular, every sesquilinear form in a real vector space is a bilinear form.*

Condition (1) *is equivalent to the following two properties*

$$(1) \quad S(f + g, h) = S(f, h) + S(g, h)$$

$$(2) \quad S(af, g) = aS(f, g)$$

Similarly, condition (2) *is equivalent to*

$$(1) \quad S(f, g + h) = S(f, g) + S(f, h)$$

$$(2) \quad S(f, ag) = \bar{a}S(f, g)$$

Definition 2. **[10]** *A sesquilinear form S on H is said to be* **Hermitian** *if for every $f, g \in H$:*

$$S(f,g) = \overline{S(g,f)}$$

Remark 1. **[10]** *A sesquilinear form S on a real vector space is automatically a bilinear form.*

Definition 3. **[11]** *A Hermitian sesquilinear form is called non-negative if*

$$S(f,f) \geq 0 \quad \text{for all } f \in H,$$

and it is called positive if

$$S(f,f) > 0 \quad \text{for all } f \in H \text{ with } f \neq 0.$$

Example 1. **[11]** *For each $(m \in \mathbb{N})(\mathbb{N} = \{1, 2, 3, \dots\})$, let $\mathbb{C}^m$ denote the complex vector space of the m-tuples $f = (f_1, f_2, \dots, f_m)$ of complex numbers, with the addition defined by*

$$f + g = (f_1 + g_1, f_2 + g_2, \dots, f_m + g_m),$$

and scalar multiplication by $a \in \mathbb{C}$ given by

$$af = (af_1, af_2, \dots, af_m).$$

If $(a_{jk})_{j,k=1,\cdots,m}$ is a complex $m \times m$ matrix, then

$$S(f,g) = \textstyle\sum_{j,k=1}^{m} a_{jk} f_j \bar{g}_k \text{ for } f, g \in \mathbb{C}^m$$

defines a sesquilinear form on $\mathbb{C}^m$.

Example 2. **[11]** *Let $C[0,1]$ denote the complex vector space of continuous functions on $[0,1]$. For a continuous functions $r : [0,1] \to \mathbb{C}$,define*

$$S(f,g) = \textstyle\int_0^1 f(x)\overline{g(x)}\, r(x)\, dx \quad f, g \in C[0,1].$$

This form is Hermitian if and only if r is real-valued. It is non-negative if and only if $r(x) \geq 0$ for all $x \in [0,1]$; and it is positive if and only if $r(x) \geq 0$ for all $x \in [0,1]$ and r does not vanish identically on any nontrivial interval.

Definition 4. **[13]** *Given a sesquilinear form S on H, the map $q : H \to \mathbb{C}$ defined by $q(f) = S(f,f)$ for each $f \in H$, is called a quadratic form on H generated by S. For each quadratic form q, $q(af) = |a|^2 q(f)$ for all $f \in H$, $a \in K$, in particular $q(af) = q(f)$ for every $a \in K$ if $|a| = 1$.*

Definition 5. [4] *A function $B : V \times V \to \mathbb{C}$ is a bilinear form on V if it is linear in each variable, when the other variable is fixed*

(1) $B(\alpha_1 v_1 + \alpha_2 v_2, w) = \alpha_1 B(v_1, w) + \alpha_2 B(v_2, w)$ *for all* $v_1, v_2, w \in V$ *and* $\alpha_1, \alpha_2, \in \mathbb{C}$

(2) $B(v, \beta_1 w_1 + \beta_2 w_2) = \beta_1 B(v, w_1) + \beta_2 B(v, w_2)$ *for all* $v, w_1, w_2 \in V$ *and* $\beta_1, \beta_2 \in \mathbb{C}$

Remark 2. [4] *An inner product on a real vector space is a bilinear form. However, an inner product on a complex vector space is not, since it is conjugate-linear in the second component rather than strictly linear.*

Example 3. [4] *For $V = C[a, b]$ the space of continuous functions, the bilinear form $\varphi(f, g) = \int_a^b f(x)g(x)dx$ is defined for all $f, g \in V$.*

Definition 6. [9] *A bilinear form B on V is symmetric if*

$$B(v, w) = B(w, v) \qquad \forall \quad v, w \in V$$

Definition 7. [9] *If B is a symmetric bilinear form on V, then $q : V \to \mathbb{C}$ given by $q(v) = B(v, v)$ is called a quadratic form generated by B.*

Example 4. [9] *If B is an inner product on a real vector space, then the quadratic form is $q(v) = \|v\|^2$.*

Theorem 1. [14] *Let H be a complex vector space, S be a sesquilinear form on H, and q the quadratic form generated by S. Then for all $f, g \in H$:*

$$S(f, g) = \tfrac{1}{4}\{q(f + g) - q(f - g) + iq(f + ig) - iq(f - ig)\}.$$

Theorem 2. [14] *Let S be a sesquilinear form on a vector space H, and let q be the corresponding quadratic form on H. Then for all $f, g \in H$:*

$$q(f + g) + q(f - g) = 2[q(f) + q(g)].$$

Theorem 3. [14] *If S is a non-negative sesquilinear form on H, and q denotes the quadratic form generated by S, then for every $f, g \in H$ the following Schwarz inequality is satisfied*

$$|S(f, g)| \leq [q(f)q(g)]^{\frac{1}{2}}.$$

3. Sesquilinear A-form

Definition 8. *Let X be a $\mathbb{C}$-vector space and A be a C^*-algebra. A left A-vector space is a $\mathbb{C}$-vector space such that for all $u, v, \in X$ and $a_1, a_2 \in A$ the following conditions are satisfied*

(1) $(a_1 + a_2)u = a_1 u + a_2 u$ *where* $a, b \in A, u \in X$

(2) $a_1(u + v) = a_1 u + a_1 v$ *where* $a \in A, u, v \in X$

(3) $1_A.u = u \in X$ *where* 1_A *is the identity element of* A

(4) $a_1(a_2)u = (a_1 a_2)u$ *where* $a, b \in A, u \in X$

Definition 9. *A left sesquilinear A-form on a vector space V over a field A is a map ${}_AS : V \times V \to A$ that satisfies*

(1) ${}_AS(v_1, w_1 + w_2) ={}_A S(v_1, w_1) +{}_A S(v_1, w_2)$

(2) ${}_AS(av_1, w_1) = a\,{}_AS(v_1, w_1)$

(3) ${}_AS(v_1, aw_1) ={}_A S(v_1, w_1)a^*$ $\forall \quad v_1, w_1, \in V, a \in A.$

Example 5. *Let A be a C^*-algebra and define ${}_AS : A \times A \to A$ by*

$${}_AS(f, g) = \lambda f g^* \qquad where \quad f, g \in A, \quad \lambda \in \mathbb{C}(\lambda \neq 0).$$

This is a left sesquilinear A-Form.

Proof.

$$(1) \quad {}_AS(f, g+h) = \lambda f(g+h)^* = \lambda f(g^*+h^*) = \lambda f g^*+\lambda f h^* ={}_A S(f, g)+{}_AS(f, h).$$

$$(2) \quad {}_AS(af, g) = \lambda(af)g^* = (\lambda a)fg^* = a(\lambda f g^*) = a\,{}_AS(f, g).$$

$$(3) \quad {}_AS(f, ag) = \lambda f(ag)^* = \lambda f(g^* a^*) = (\lambda f g^*)a^* ={}_A S(f, g)a^*.$$

Example 6. *Let $X = \mathbb{C}^n$, $A = M_n(\mathbb{C}^n)$*

$${}_AS(x, y) = xy^* \qquad\qquad where\ y^* = (\bar{y})^\top$$

Proof. *We show that* ${}_AS(ax,z) = a_A S(x,z)$

L.H.S: ${}_AS(ax,z) = (ax)(\bar{z})^\top$

$$ax = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^n a_{1j}x_j \\ \sum_{j=1}^n a_{2j}x_j \\ \vdots \\ \sum_{j=1}^n a_{nj}x_j \end{pmatrix}$$

$$(ax)\bar{z}^\top = \begin{pmatrix} \sum_{j=1}^n a_{1j}x_j \\ \sum_{j=1}^n a_{2j}x_j \\ \vdots \\ \sum_{j=1}^n a_{nj}x_j \end{pmatrix} \begin{pmatrix} \bar{z}_1 & \bar{z}_2 & \cdots & \bar{z}_n \end{pmatrix} = \begin{pmatrix} (\sum_{j=1}^n a_{1j}x_j)\bar{z}_1 & (\sum_{j=1}^n a_{1j}x_j)\bar{z}_2 & \cdots & (\sum_{j=1}^n a_{1j}x_j)\bar{z}_n \\ (\sum_{j=1}^n a_{2j}x_j)\bar{z}_1 & (\sum_{j=1}^n a_{2j}x_j)\bar{z}_2 & \cdots & (\sum_{j=1}^n a_{2j}x_j)\bar{z}_n \\ \vdots & \vdots & \vdots & \vdots \\ (\sum_{j=1}^n a_{nj}x_j)\bar{z}_1 & (\sum_{j=1}^n a_{nj}x_j)\bar{z}_2 & \cdots & (\sum_{j=1}^n a_{nj}x_j)\bar{z}_n \end{pmatrix}$$

R.H.S= $a_A S(x,z) = axz^* = ax\bar{z}^\top$

$$x\bar{z}^\top = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \begin{pmatrix} \bar{z}_1 & \bar{z}_2 & \cdots & \bar{z}_n \end{pmatrix} = \begin{pmatrix} x_1\bar{z}_1 & x_1\bar{z}_2 & \cdots & x_1\bar{z}_n \\ x_2\bar{z}_1 & x_2\bar{z}_2 & \cdots & x_2\bar{z}_n \\ \vdots & \vdots & \vdots & \vdots \\ x_n\bar{z}_1 & x_n\bar{z}_2 & \cdots & x_n\bar{z}_n \end{pmatrix}$$

$$a(x\bar{z}^\top) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1\bar{z}_1 & x_1\bar{z}_2 & \cdots & x_1\bar{z}_n \\ x_2\bar{z}_1 & x_2\bar{z}_2 & \cdots & x_2\bar{z}_n \\ \vdots & \vdots & \vdots & \vdots \\ x_n\bar{z}_1 & x_n\bar{z}_2 & \cdots & x_n\bar{z}_n \end{pmatrix}$$

$$a(x\bar{z}^\top) = \begin{pmatrix} (\sum_{j=1}^n a_{1j}x_j)\bar{z}_1 & (\sum_{j=1}^n a_{1j}x_j)\bar{z}_2 & \cdots & (\sum_{j=1}^n a_{1j}x_j)\bar{z}_n \\ (\sum_{j=1}^n a_{2j}x_j)\bar{z}_1 & (\sum_{j=1}^n a_{2j}x_j)\bar{z}_2 & \cdots & (\sum_{j=1}^n a_{2j}x_j)\bar{z}_n \\ \vdots & \vdots & \vdots & \vdots \\ (\sum_{j=1}^n a_{nj}x_j)\bar{z}_1 & (\sum_{j=1}^n a_{nj}x_j)\bar{z}_2 & \cdots & (\sum_{j=1}^n a_{nj}x_j)\bar{z}_n \end{pmatrix}$$

Hence, the equality holds. $\Rightarrow_A S(ax,z) = a_A S(x,z)$

We also need to prove that ${}_AS(x,az) =_A S(x,z)a^*$

L.H.S: ${}_AS(x,az) = x(az)^*$

$$az = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^n a_{1j}z_j \\ \sum_{j=1}^n a_{2j}z_j \\ \vdots \\ \sum_{j=1}^n a_{nj}z_j \end{pmatrix}$$

$$(az)^* = (\bar{az})^\top = \begin{pmatrix} \sum_{j=1}^n \overline{a_{1j}z_j} & \sum_{j=1}^n \overline{a_{2j}z_j} & \cdots & \sum_{j=1}^n \overline{a_{nj}z_j} \end{pmatrix}$$

Then

$$x(az)^* = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \begin{pmatrix} \sum_{j=1}^n \overline{a_{1j}z_j} & \sum_{j=1}^n \overline{a_{2j}z_j} & \cdots & \sum_{j=1}^n \overline{a_{nj}z_j} \end{pmatrix}$$

$$= \begin{pmatrix} (\sum_{j=1}^n \overline{a_{1j}z_j})x_1 & (\sum_{j=1}^n \overline{a_{2j}z_j})x_1 & \cdots & (\sum_{j=1}^n \overline{a_{nj}z_j})x_1 \\ (\sum_{j=1}^n \overline{a_{1j}z_j})x_2 & (\sum_{j=1}^n \overline{a_{2j}z_j})x_2 & \cdots & (\sum_{j=1}^n \overline{a_{nj}z_j})x_2 \\ \vdots & \vdots & \vdots & \vdots \\ (\sum_{j=1}^n \overline{a_{1j}z_j})x_n & (\sum_{j=1}^n \overline{a_{2j}z_j})x_n & \cdots & (\sum_{j=1}^n \overline{a_{nj}z_j})x_n \end{pmatrix}$$

$R.H.S :=_A S(x,z)a^*$

$$xz^* = x(\bar{z})^\top = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \begin{pmatrix} \bar{z_1} & \bar{z_2} & \cdots & \bar{z_n} \end{pmatrix} = \begin{pmatrix} x_1\bar{z}_1 & x_1\bar{z}_2 & \cdots & x_1\bar{z}_n \\ x_2\bar{z}_1 & x_2\bar{z}_2 & \cdots & x_2\bar{z}_n \\ \vdots & \vdots & \vdots & \vdots \\ x_n\bar{z}_1 & x_n\bar{z}_2 & \cdots & x_n\bar{z}_n \end{pmatrix}$$

$$x(\bar{z})^\top (a)^* = \begin{pmatrix} x_1\bar{z}_1 & x_1\bar{z}_2 & \cdots & x_1\bar{z}_n \\ x_2\bar{z}_1 & x_2\bar{z}_2 & \cdots & x_2\bar{z}_n \\ \vdots & \vdots & \vdots & \vdots \\ x_n\bar{z}_1 & x_n\bar{z}_2 & \cdots & x_n\bar{z}_n \end{pmatrix} \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & \vdots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{pmatrix}$$

$$= \begin{pmatrix} (\sum_{j=1}^n \overline{a_{1j}z_j})x_1 & (\sum_{j=1}^n \overline{a_{2j}z_j})x_1 & \cdots & (\sum_{j=1}^n \overline{a_{nj}z_j})x_1 \\ (\sum_{j=1}^n \overline{a_{1j}z_j})x_2 & (\sum_{j=1}^n \overline{a_{2j}z_j})x_2 & \cdots & (\sum_{j=1}^n \overline{a_{nj}z_j})x_2 \\ \vdots & \vdots & \vdots & \vdots \\ (\sum_{j=1}^n \overline{a_{1j}z_j})x_n & (\sum_{j=1}^n \overline{a_{2j}z_j})x_n & \cdots & (\sum_{j=1}^n \overline{a_{nj}z_j})x_n \end{pmatrix}$$

Then we find that ${}_AS(x,az) =_A S(x,z)a^*$

Definition 10. *A right sesquilinear A-form on a vector space V over a field A is a map $S_A : V \times V \to A$ satisfying:*

(1) $S_A(v_1, w_1 + w_2) = S_A(v_1, w_1) + S_A(v_1, w_2)$

(2) $S_A(v_1 a, w_1) = a^* S_A(v_1, w_1)$

(3) $S_A(v_1, w_1 a) = S_A(v_1, w_1)a$ *for all* $v_1, w_1, \in V, a \in A.$

Example 7. *Let A be a C^*-algebra and define $S_A : A \times A \to A$ by*

$$S_A(a,b) = \lambda a^* b \qquad where \quad a,b \in A, \lambda \in \mathbb{C}(\lambda \neq 0).$$

This is a right sesquilinear A-Form.

Proof.

$$(1) \quad S_A(f, g+h) = \lambda f^*(g+h) = \lambda(f^*g + f^*h) = \lambda f^*g + \lambda f^*h = S_A(f,g) + S_A(f,h).$$

$$(2) \quad S_A(fa, g) = \lambda(fa)^*g = \lambda a^* f^* g = a^*(\lambda f^* g) = a^* S_A(f,g).$$

$$(3) \quad S_A(f, ga) = \lambda f^*(ga) = \lambda(f^*g)a == S_A(f,g)a.$$

Definition 11. *A sesquilinear A-form on a vector space V over a field A is a map* ${}_AS_A : V \times V \to A$ *which satisfies:*

(1) ${}_AS_A(cv_1, w_1) = c{}_AS_A(v_1, w_1)$ *and* ${}_AS_A(v_1, cw_1) ={}_A S_A(v_1, w_1)c^*$

(2) ${}_AS_A(v_1c, w_1) = c^*{}_AS_A(v_1, w_1)$ *and* ${}_AS_A(v_1, w_1c) ={}_A S_A(v_1, w_1)c$

(3) ${}_AS_A(v_1 + v_2, w_1) ={}_A S_A(v_1, w_1) +{}_A S_A(v_2, w_1)$ *and* ${}_AS_A(v_1, w_1+w_2) ={}_A S_A(v_1, w_1)+{}_AS_A(v_1, w_2)$ *for all* $v_1, v_2, w_1, w_2 \in V, c \in A$.

Example 8. *Let A be a left and right sesquilinear A-Form. If* $S_A : A \times A \to A$ *is defined by*

$$S_A(a,b) = \lambda a^* b \qquad {}_AS(a,b) = \lambda ab^* \qquad where \quad a,b \in A, \quad \lambda \in \mathbb{C}(\lambda \neq 0).$$

Then, this is a sesquilinear A-Form.

Proof.

$$(1) \quad {}_AS(af, g) = \lambda(af)g^* = (\lambda a)fg^* = a(\lambda fg^*) = a{}_AS(f,g)$$

$${}_AS(f, ag) = \lambda f(ag)^* = \lambda f(g^*a^*) = (\lambda fg^*)a^* ={}_A S(f,g)a^*.$$

$$(2) \quad S_A(fa, g) = \lambda(fa)^*g = \lambda a^* f^* g = a^*(\lambda f^* g) = a^* S_A(f,g).$$

$$S_A(f, ga) = \lambda f^*(ga) = \lambda(f^*g)a == S_A(f,g)a.$$

$$(3) \quad S_A(f, g+h) = \lambda f^*(g+h) = \lambda(f^*g + f^*h) = \lambda f^*g + \lambda f^*h \\ = S_A(f,g) + S_A(f,h).$$

$$S_A(f+g, h) = \lambda(f+g)^*h = \lambda(f^*+g^*)h = \lambda f^*h + \lambda g^*h = S_A(f,h) + S_A(g,h).$$

$$_AS(f, g+h) = \lambda f(g+h)^* = \lambda f(g^*+h^*) = \lambda fg^* + \lambda fh^* =_A S(f,g) +_A S(f,h).$$

$$_AS(f+g, h) = \lambda(f+g)h^* = \lambda(fh^*+gh^*) = \lambda fh^* + \lambda gh^* =_A S(f,h) +_A S(g,h).$$

Definition 12. *A right bilinear A-form on a vector space X over a filed A is a map $B_A : X \times X \to A$ which satisfies the following:*

(1) $B_A(ax + by, z) = aB_A(x,y) + bB_A(x,z)$

(2) $B_A(x, ya + zb) = B_A(x,y)a + B_A(x,z)b.$

This means that a bilinear A-form is left A-linear in the first and it is right A-linear in the second.

Example 9. *let $X = \mathbb{C}^n$, $A = M_n(\mathbb{C}^n)$*

$$B_A(x,y) = xy.$$

Proof. We show that $B_A(ax,y) = aB_A(x,y)$

L.H.S:$B_A(ax,y) = axy$

$$ax = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^n a_{1j}x_{j1} & \sum_{j=1}^n a_{1j}x_{j2} & \cdots & \sum_{j=1}^n a_{1j}x_{jn} \\ \sum_{j=1}^n a_{2j}x_{j1} & \sum_{j=1}^n a_{2j}x_{j2} & \cdots & \sum_{j=1}^n a_{2j}x_{jn} \\ \vdots \\ \sum_{j=1}^n a_{nj}x_{j1} & \sum_{j=1}^n a_{nj}x_{j2} & \cdots & \sum_{j=1}^n a_{nj}x_{jn} \end{pmatrix}$$

$$(ax)y = \begin{pmatrix} \sum_{j=1}^n a_{1j}x_{j1} & \sum_{j=1}^n a_{1j}x_{j2} & \cdots & \sum_{j=1}^n a_{1j}x_{jn} \\ \sum_{j=1}^n a_{2j}x_{j1} & \sum_{j=1}^n a_{2j}x_{j2} & \cdots & \sum_{j=1}^n a_{2j}x_{jn} \\ \vdots \\ \sum_{j=1}^n a_{nj}x_{j1} & \sum_{j=1}^n a_{nj}x_{j2} & \cdots & \sum_{j=1}^n a_{nj}x_{jn} \end{pmatrix} \begin{pmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ y_{21} & y_{22} & \cdots & y_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ y_{n1} & y_{n2} & \cdots & y_{nn} \end{pmatrix}$$

$$axy = \begin{pmatrix} \sum_{j=1}^{n} a_{1j}y_{j1}\sum_{j=1}^{n}(\sum_{i=1}^{n} x_{ji}) & \sum_{j=1}^{n} a_{1j}y_{j2}\sum_{j=1}^{n}(\sum_{i=1}^{n} x_{ji}) & \cdots & \sum_{j=1}^{n} a_{1j}y_{jn}\sum_{j=1}^{n}(\sum_{i=1}^{n} x_{ji}) \\ \sum_{j=1}^{n} a_{2j}y_{j1}\sum_{j=1}^{n}(\sum_{i=1}^{n} x_{ji}) & \sum_{j=1}^{n} a_{2j}y_{j2}\sum_{j=1}^{n}(\sum_{i=1}^{n} x_{ji}) & \cdots & \sum_{j=1}^{n} a_{2j}y_{jn}\sum_{j=1}^{n}(\sum_{i=1}^{n} x_{ji}) \\ \vdots & & & \\ \sum_{j=1}^{n} a_{nj}y_{j1}\sum_{j=1}^{n}(\sum_{i=1}^{n} x_{ji}) & \sum_{j=1}^{n} a_{nj}y_{j2}\sum_{j=1}^{n}(\sum_{i=1}^{n} x_{ji}) & \cdots & \sum_{j=1}^{n} a_{nj}y_{jn}\sum_{j=1}^{n}(\sum_{i=1}^{n} x_{ji}) \end{pmatrix}$$

R.H.S$= aB_A(x, y)$

$$B_A(x, y) = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{pmatrix} \begin{pmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ y_{21} & y_{22} & \cdots & y_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ y_{n1} & y_{n2} & \cdots & y_{nn} \end{pmatrix}$$

$$B_A(x, y) = \begin{pmatrix} \sum_{j=1}^{n} x_{1j}y_{j1} & \sum_{j=1}^{n} x_{1j}y_{j2} & \cdots & \sum_{j=1}^{n} x_{1j}y_{jn} \\ \sum_{j=1}^{n} x_{2j}y_{j1} & \sum_{j=1}^{n} x_{2j}y_{j2} & \cdots & \sum_{j=1}^{n} x_{2j}y_{jn} \\ \vdots & & & \\ \sum_{j=1}^{n} x_{nj}y_{j1} & \sum_{j=1}^{n} x_{nj}y_{j2} & \cdots & \sum_{j=1}^{n} x_{nj}y_{jn} \end{pmatrix}$$

$$aB_A(x, y) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} \sum_{j=1}^{n} x_{1j}y_{j1} & \sum_{j=1}^{n} x_{1j}y_{j2} & \cdots & \sum_{j=1}^{n} x_{1j}y_{jn} \\ \sum_{j=1}^{n} x_{2j}y_{j1} & \sum_{j=1}^{n} x_{2j}y_{j2} & \cdots & \sum_{j=1}^{n} x_{2j}y_{jn} \\ \vdots & & & \\ \sum_{j=1}^{n} x_{nj}y_{j1} & \sum_{j=1}^{n} x_{nj}y_{j2} & \cdots & \sum_{j=1}^{n} x_{nj}y_{jn} \end{pmatrix}$$

Then

$$aB_A(x,y) = \begin{pmatrix} \sum_{j=1}^{n} a_{1j}y_{j1}\sum_{j=1}^{n}(\sum_{i=1}^{n} x_{ji}) & \sum_{j=1}^{n} a_{1j}y_{j2}\sum_{j=1}^{n}(\sum_{i=1}^{n} x_{ji}) & \cdots & \sum_{j=1}^{n} a_{1j}y_{jn}\sum_{j=1}^{n}(\sum_{i=1}^{n} x_{ji}) \\ \sum_{j=1}^{n} a_{2j}y_{j1}\sum_{j=1}^{n}(\sum_{i=1}^{n} x_{ji}) & \sum_{j=1}^{n} a_{2j}y_{j2}\sum_{j=1}^{n}(\sum_{i=1}^{n} x_{ji}) & \cdots & \sum_{j=1}^{n} a_{2j}y_{jn}\sum_{j=1}^{n}(\sum_{i=1}^{n} x_{ji}) \\ \vdots \\ \sum_{j=1}^{n} a_{nj}y_{j1}\sum_{j=1}^{n}(\sum_{i=1}^{n} x_{ji}) & \sum_{j=1}^{n} a_{nj}y_{j2}\sum_{j=1}^{n}(\sum_{i=1}^{n} x_{ji}) & \cdots & \sum_{j=1}^{n} a_{nj}y_{jn}\sum_{j=1}^{n}(\sum_{i=1}^{n} x_{ji}) \end{pmatrix}$$

Then we get $B_A(ax,y) = aB_A(x,y)$

We also need to prove that $B_A(x,ya) = B_A(x,y)a$

L.H.S:= $B_A(x,ya) = xya$

$$xy = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{pmatrix} \begin{pmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ y_{21} & y_{22} & \cdots & y_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ y_{n1} & y_{n2} & \cdots & y_{nn} \end{pmatrix}$$

$$xy = \begin{pmatrix} \sum_{j=1}^{n} x_{1j}y_{j1} & \sum_{j=1}^{n} x_{1j}y_{j2} & \cdots & \sum_{j=1}^{n} x_{1j}y_{jn} \\ \sum_{j=1}^{n} x_{2j}y_{j1} & \sum_{j=1}^{n} x_{2j}y_{j2} & \cdots & \sum_{j=1}^{n} x_{2j}y_{jn} \\ \vdots \\ \sum_{j=1}^{n} x_{nj}y_{j1} & \sum_{j=1}^{n} x_{nj}y_{j2} & \cdots & \sum_{j=1}^{n} x_{nj}y_{jn} \end{pmatrix}$$

$$xya = \begin{pmatrix} \sum_{j=1}^{n} x_{1j}y_{j1} & \sum_{j=1}^{n} x_{1j}y_{j2} & \cdots & \sum_{j=1}^{n} x_{1j}y_{jn} \\ \sum_{j=1}^{n} x_{2j}y_{j1} & \sum_{j=1}^{n} x_{2j}y_{j2} & \cdots & \sum_{j=1}^{n} x_{2j}y_{jn} \\ \vdots \\ \sum_{j=1}^{n} x_{nj}y_{j1} & \sum_{j=1}^{n} x_{nj}y_{j2} & \cdots & \sum_{j=1}^{n} x_{nj}y_{jn} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

$$xya = \begin{pmatrix} \sum_{j=1}^{n} x_{1j}a_{j1}\sum_{j=1}^{n}(\sum_{i=1}^{n} y_{ji}) & \sum_{j=1}^{n} x_{1j}a_{j2}\sum_{j=1}^{n}(\sum_{i=1}^{n} y_{ji}) & \cdots & \sum_{j=1}^{n} x_{1j}a_{jn}\sum_{j=1}^{n}(\sum_{i=1}^{n} y_{ji}) \\ \sum_{j=1}^{n} x_{2j}a_{j1}\sum_{j=1}^{n}(\sum_{i=1}^{n} y_{ji}) & \sum_{j=1}^{n} x_{2j}a_{j2}\sum_{j=1}^{n}(\sum_{i=1}^{n} y_{ji}) & \cdots & \sum_{j=1}^{n} x_{2j}a_{jn}\sum_{j=1}^{n}(\sum_{i=1}^{n} y_{ji}) \\ \vdots \\ \sum_{j=1}^{n} x_{nj}a_{j1}\sum_{j=1}^{n}(\sum_{i=1}^{n} y_{ji}) & \sum_{j=1}^{n} x_{nj}a_{j2}\sum_{j=1}^{n}(\sum_{i=1}^{n} y_{ji}) & \cdots & \sum_{j=1}^{n} x_{nj}a_{jn}\sum_{j=1}^{n}(\sum_{i=1}^{n} y_{ji}) \end{pmatrix}$$

R.H.S= $B_A(x, y)a$

$$B_A(x, y) = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{pmatrix} \begin{pmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ y_{21} & y_{22} & \cdots & y_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ y_{n1} & y_{n2} & \cdots & y_{nn} \end{pmatrix}$$

$$B_A(x, y) = \begin{pmatrix} \sum_{j=1}^{n} x_{1j}y_{j1} & \sum_{j=1}^{n} x_{1j}y_{j2} & \cdots & \sum_{j=1}^{n} x_{1j}y_{jn} \\ \sum_{j=1}^{n} x_{2j}y_{j1} & \sum_{j=1}^{n} x_{2j}y_{j2} & \cdots & \sum_{j=1}^{n} x_{2j}y_{jn} \\ \vdots \\ \sum_{j=1}^{n} x_{nj}y_{j1} & \sum_{j=1}^{n} x_{nj}y_{j2} & \cdots & \sum_{j=1}^{n} x_{nj}y_{jn} \end{pmatrix}$$

$$B_A(x, y)a = \begin{pmatrix} \sum_{j=1}^{n} x_{1j}y_{j1} & \sum_{j=1}^{n} x_{1j}y_{j2} & \cdots & \sum_{j=1}^{n} x_{1j}y_{jn} \\ \sum_{j=1}^{n} x_{2j}y_{j1} & \sum_{j=1}^{n} x_{2j}y_{j2} & \cdots & \sum_{j=1}^{n} x_{2j}y_{jn} \\ \vdots \\ \sum_{j=1}^{n} x_{nj}y_{j1} & \sum_{j=1}^{n} x_{nj}y_{j2} & \cdots & \sum_{j=1}^{n} x_{nj}y_{jn} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

$$B_A(x, y)a = \begin{pmatrix} \sum_{j=1}^{n} x_{1j}a_{j1}\sum_{j=1}^{n}(\sum_{i=1}^{n} y_{ji}) & \sum_{j=1}^{n} x_{1j}a_{j2}\sum_{j=1}^{n}(\sum_{i=1}^{n} y_{ji}) & \cdots & \sum_{j=1}^{n} x_{1j}a_{jn}\sum_{j=1}^{n}(\sum_{i=1}^{n} y_{ji}) \\ \sum_{j=1}^{n} x_{2j}a_{j1}\sum_{j=1}^{n}(\sum_{i=1}^{n} y_{ji}) & \sum_{j=1}^{n} x_{2j}a_{j2}\sum_{j=1}^{n}(\sum_{i=1}^{n} y_{ji}) & \cdots & \sum_{j=1}^{n} x_{2j}a_{jn}\sum_{j=1}^{n}(\sum_{i=1}^{n} y_{ji}) \\ \vdots \\ \sum_{j=1}^{n} x_{nj}a_{j1}\sum_{j=1}^{n}(\sum_{i=1}^{n} y_{ji}) & \sum_{j=1}^{n} x_{nj}a_{j2}\sum_{j=1}^{n}(\sum_{i=1}^{n} y_{ji}) & \cdots & \sum_{j=1}^{n} x_{nj}a_{jn}\sum_{j=1}^{n}(\sum_{i=1}^{n} y_{ji}) \end{pmatrix}$$

Thus we obtain that $B_A(x, ya) = B_A(x, y)a$

Definition 13. *Let $_AS$ be a left sesquilinear A-form on H. A map $q_A : H \to A$ defined by $q_A(x) =_A S(x, x)$ for each $x \in H$ is called the quadratic A-form introduced by $_AS$.*

Theorem 4. *Let $_AS$ be a left sesquilinear A-form on a vector space H, and let q_A be the corresponding quadratic A-form on H. Then for all $\alpha, \beta \in H$*

$$\|_AS(\alpha, \beta)\| \leq [q_A(\alpha) \cdot q_A(\beta)]^{\frac{1}{2}}$$

Proof.

$$0_A \leq q_A(\alpha - c_A\beta) =_A S(\alpha - c_A\beta, \alpha - c_A\beta)$$

$$=_A S(\alpha, \alpha) +_A S(\alpha, -c_A\beta) +_A S(-c_A\beta, \alpha) +_A S(-c_A\beta, -c_A\beta)$$

Then

$$0_A \leq_A S(\alpha, \alpha) +_A S(\alpha, \beta)(-c_A)^* + (-c_A)_AS(\beta, \alpha) + (-c_A)_AS(\beta, \beta)(-c_A)^*$$

$$= q_A(\alpha) +_A S(\alpha, \beta)(-c_A)^* + (-c_A)_AS(\beta, \alpha) + (-c_A)q_A(\beta)(-c_A)^*$$

$$= q_A(\alpha) +_A S(\alpha, \beta)(c_A)^*(-1)^*(-c_A)_AS(\beta, \alpha) - c_Aq_A(\beta)(c_A)^*(-1)^*$$

$$= q_A(\alpha) -_A S(\alpha, \beta)(c_A)^*(-c_A)_AS(\beta, \alpha) + c_Aq_A(\beta)(c_A)^*$$

Let $c_A = \frac{_AS(\alpha,\beta)}{q_A(\beta)}$.

$$0_A \leq q_A(\alpha) -_A S(\alpha, \beta)(\tfrac{_AS(\alpha,\beta)}{q_A(\beta)})^* - \tfrac{_AS(\alpha,\beta)}{q_A(\beta)}{}_AS(\beta, \alpha) + \tfrac{_AS(\alpha,\beta)}{q_A(\beta)}q_A(\beta)(\tfrac{_AS(\alpha,\beta)}{q_A(\beta)})^*$$

$$= q_A(\alpha) - \tfrac{_AS(\alpha,\beta)(_AS(\alpha,\beta))^*}{(q_A(\beta))^*} - \tfrac{_AS(\alpha,\beta)(_AS(\alpha,\beta))^*}{q_A(\beta)} + \tfrac{_AS(\alpha,\beta)(_AS(\alpha,\beta))^*}{(q_A(\beta))^*}$$

$$= q_A(\alpha) - \tfrac{\|_AS(\alpha,\beta)\|^2}{q_A^*(\beta)} - \tfrac{\|_AS(\alpha,\beta)\|^2}{q_A(\beta)} + - \tfrac{\|_AS(\alpha,\beta)\|^2}{q_A^*(\beta)}.$$

Then

$\frac{\|_AS(\alpha,\beta)\|^2}{q_A(\beta)} \leq q_A(\alpha)$ which gives the required result $\|_AS(\alpha, \beta)\| \leq [q_A(\alpha) \cdot q_A(\beta)]^{\frac{1}{2}}$.

Proposition 1. *A mapping $_AS : V \longrightarrow A$ is called a semi quadratic A-form on an A-vector space V, if for all $f, g \in V$ and $a \in A$, the following hold:*

(1) $p_A(af) \leq \|a\|p_A(f)$

Proof.

$since\ p_A(f) = \sqrt{\|_AS(f,f)\|}$

$$p_A^2(af) = \|_AS(af,af)\| = \|a(_AS(f,f))a^*\| = \|a(_AS^{\frac{1}{2}}(f,f))_AS^{\frac{1}{2}}(f,f)a^*\|$$

$$= \|(a(_AS^{\frac{1}{2}}(f,f)))(a(_AS^{\frac{1}{2}}(f,f)))^*\| = \|a(_AS_A^{\frac{1}{2}}(f,f))\|^2$$

$$\Rightarrow p_A(af) = \|a(_AS^{\frac{1}{2}}(f,f))\| \Rightarrow p_A(af) \leq \|a\|\|a(_AS^{\frac{1}{2}}(f,f))S^{\frac{1}{2}}(f,f)\|$$

$$\Rightarrow p_A(af) \leq \|a\|p_A(f).$$

(2) $p_A(f+g) \leq p_A(f) + p_A(g);$ $\qquad {}_AS(g,f) = \overline{{}_AS(f,g)}$

Proof.

$$p_A^2(f+g) = \|_AS(f+g,f+g)\| = \|_AS(f,f)+_AS(f,g)+_AS(g,f)+_AS(g,g)\|$$

$$= \|_AS(f,f)+_AS(f,g)+\overline{_AS(f,g)}+_AS(g,g)\|$$

$$= \|_AS(f,f) + +2Re_AS(f,g) +_A S(g,g)\|$$

$$\leq \|_AS(f,f)\| + \|_AS(g,g)\| + 2\|_AS(f,g)\|$$

$$= \|_AS(f,f)\| + \|_AS(g,g)\| + 2(p_A(f)\cdot p_A(g))^{\frac{1}{2}}$$

$$\leq \|_AS(f,f)\| + \|_AS(g,g)\| + p_A(f)\cdot p_A(g)$$

$$= p_A^2(f)+p_A^2(g)+2p_A(f)\cdot p_A(g) = (p_A(f)+p_A(g))^2$$

$$\Rightarrow p_A(f+g) \leq p_A(f) + p_A(g).$$

(3) $p_A(f) > 0\ for\ all\ f \neq 0$

Proof.

$Since\ p_A(f) = \sqrt{\|_AS(f,f)\|} > 0\ then\ p_A(f) > 0$

Proposition 2. *If p_A is a semi-quadratic A form on V, then for all $f, g \in V$ and $a \in A$, then*

$$p_A(f-g) \geq \|p_A(f) - p_A(g)\|.$$

Proof. Using the triangle inequality:

$$p_A(f) = p_A(f-g+g) \leq p_A(f-g) + p_A(g)$$

So that

$$p_A(f) - p_A(g) \leq p_A(f-g)$$

Similarly, $p_A(g) = p_A(g-f+f) \leq p_A(g-f) + p_A(f);$ *which yields*

$$-(p_A(f) - p_A(g)) \leq p_A(g-f)$$

Thus, $p_A(f-g) \geq \|p_A(f) - p_A(g)\|$. *Also by a similar method we can proof the inequality* $p_A(f+g) \geq \|p_A(f) - p_A(g)\|$

Proposition 3. *If p_A is a semi quadratic A-form on V, then for all $f, g \in V$ and $a \in A$:*

$$p_A^2(f+g) + p_A^2(f-g) \leq 2(p_A(f) + p_A(g))^2.$$

Proof.

$$\begin{aligned}
p_A^2(f+g) + p_A^2(f-g) &= \|{}_AS(f+g, f+g)\| + \|{}_AS(f-g, f-g)\| \\
&= \|{}_AS(f,f) +_A S(f,g) +_A S(g,f) +_A S(g,g)\| \\
&\quad +\|{}_AS(f,f) +{}_AS(f,-g) +{}_AS(-g,f) +{}_AS(-g,-g)\| \\
&\leq \|{}_AS(f,f)\| + \|{}_AS(f,g)\| + \|{}_AS(g,f)\| + \|{}_AS(g,g)\| \\
&\quad +\|{}_AS(f,f)\| + \|{}_AS(f,g)(-1)^*\| + \|(-1){}_AS(g,f)\| \\
&\quad +\|(-1){}_AS(g,g)(-1)^*\| \\
&= \|{}_AS(f,f)\| + \|{}_AS(f,g)\| + \|({}_AS(f,g))^*\| + \|{}_AS(g,g)\| \\
&\quad +\|{}_AS(f,f)\| + \|{}_AS(f,g)\| + \|({}_AS(f,g))^*\| + \|{}_AS(g,g)\| \\
&= 2\|{}_AS(f,f)\| + 2\|{}_AS(g,g)\| + 4\|{}_AS(f,g)\|
\end{aligned}$$

$$= 2[q_A^2(f) + q_A^2(g) + 2q_A(f) \cdot q_A(f)] = 2[q_A(f) + q_A(g)]^2$$

$$= 2[p_A(f) + p_A(g)]^2 \ .$$

Hence, $p_A^2(f+g) + p_A^2(f-g) \leq 2(p_A(f) + p_A(g))^2$.

Proposition 4. *Let* p_A *be a semi quadratic* A*-form on* V *and* ${}_AS$ *be a left sesquilinear* A*-form. Then for all* $f, g \in V$*:*

$$p_A^2(f+g) + ip_A^2(f+ig) \leq 4[p_A^2(f-g) + ip_A^2(f-ig)]$$

Proof.

$$\tfrac{1}{4}[p_A^2(f+g) + ip_A^2(f+ig) + p_A^2(f-g) + ip_A^2(f-ig)]$$

$$= \tfrac{1}{4}[\|{}_AS(f+g, f+g)\| - \|{}_AS(f-g, f-g)\|$$

$$+i\|{}_AS(f+ig, f+ig)\| - i\|{}_AS(f-ig, f-ig)\|]$$

$$= \tfrac{1}{4}[\|{}_AS(f,f) +_A S(f,g) +_A S(g,f) +_A S(g,g)\|$$

$$-\|{}_AS(f,f) +_A S(f,-g) +_A S(-g,f) +_A S(-g,-g)\|$$

$$+i\|{}_AS(f,f) +_A S(f,ig) +_A S(ig,f) +_A S(ig,ig)\|$$

$$-i\|{}_AS(f,f) +_A S(f,-ig) +_A S(-ig,f) +_A S(-ig,-ig)\|]$$

$$= \tfrac{1}{4}[\|{}_AS(f,f) +_A S(f,g) +_A S(g,f) +_A S(g,g)\|$$

$$-\|{}_AS(f,f) +_A S(f,g)(-1)^* + (-1)_AS(g,f) + (-1)_AS(g,g)(-1)^*\|$$

$$+i\|{}_AS(f,f) +_A S(f,g)(i)^* + (i)_AS(g,f) + (i)_AS(g,g)(i)^*\|$$

$$-i\|{}_AS(f,f) +_A S(f,g)(-i)^* + (-i)_AS(g,f)$$

$$+(-i)_AS(g,g)(-i)^*\|]$$

$$\leq \tfrac{1}{4}[\|{}_AS(f,f)\| + \|{}_AS(f,g)\| + \|{}_AS(g,f)\| + \|{}_AS(g,g)\| - \|{}_AS(f,f)\|$$

$$-\|{}_AS(f,g)\| - \|{}_AS(g,f)\| - \|{}_AS(g,g)\| + i\|{}_AS(f,f)\|$$

$$+i\|_AS(f,g)\|i+i^2\|_AS(g,f)\|+i^2\|_AS(g,g)\|i-i\|_AS(f,f)\|$$

$$-i\|_AS(f,g)\|i-i^2\|_AS(g,f)\|-i^2\|_AS(g,g)\|i]\leq 0_A$$

Thus we obtain: $p_A^2(f+g)+ip_A^2(f+ig)\leq 4[p_A^2(f-g)+ip_A^2(f-ig)]$

Using the same approach, we can prove that $\frac{1}{4}[p_A^2(f+g)-p_A^2(f-g)+ip_A^2(f+ig)-ip_A^2(f-ig)]^*\leq 0_A$

Proposition 5. *Let* p_A *be a semi quadratic A-form on* V *and* ${}_AS$ *be a left sesquilinear A-form. Then for all* $f\in V$*:*

$$\tfrac{1}{4}[p_A^2(f+f)-p_A^2(f-f)+ip_A^2(f+if)-ip_A^2(f-if)]=p_A^2(f+f)$$

Proof.

$$\tfrac{1}{4}[p_A^2(f+f)-p_A^2(f-f)+ip_A^2(f+if)++ip_A^2(f-if)]$$

$$=\tfrac{1}{4}[\|_AS(f+f,f+f)\|-\|_AS(f-f,f-f)\|$$

$$+i\|_AS(f+if,f+if)\|-i\|_AS(f-if,f-if)\|]$$

$$=\tfrac{1}{4}[\|_AS(f,f)+_AS(f,f)+_AS(f,f)+_AS(f,f)\|$$

$$-\|_AS(f,f)+_AS(f,-f)+_AS(-f,f)+_AS(-f,-f)\|$$

$$+i\|_AS(f,f)+_AS(f,if)+_AS(if,f)+_AS(if,if)\|$$

$$-i\|_AS(f,f)+_AS(f,-if)+_AS(-if,f)+_AS(-if,-if)\|]$$

$$=\tfrac{1}{4}[\|_AS(f,f)+_AS(f,f)+_AS(f,f)+_AS(f,f)\|$$

$$-\|_AS(f,f)+_AS(f,f)(-1)^*+(-1)_AS(f,f)+(-1)_AS(f,f)(-1)^*\|$$

$$+i\|_AS(f,f)+_AS(f,f)(i)^*+(i)_AS(f,f)+(i)_AS(f,f)(i)^*\|$$

$$-i\|_AS(f,f)+_AS(f,f)(-i)^*+(-i)_AS(f,f)+(-i)_AS(f,f)(-i)^*\|]$$

$$=\tfrac{1}{4}[\|4_AS(f,f)\|-\|_AS(f,f)+_AS(f,f)(-1)^*-_AS(f,f)-_AS(f,f)(-1)^*\|$$

$$+i\|_AS(f,f)+_AS(f,f)(i)^*+i(_AS(f,f))+i(_AS(f,f))(i)^*\|$$

$$-i\|_AS(f,f)+_AS(f,f)(-i)^*-i(_AS(f,f))-i(_AS(f,f))(-i)^*\|]$$

$$=\frac{1}{4}[\|4_AS(f,f)\|+i\|_AS(f,f)-_AS(f,f)i+i(_AS(f,f))-i(_AS(f,f))i\|$$

$$-i\|_AS(f,f)+_AS(f,f)i-i(_AS(f,f))-i(_AS(f,f))i\|]$$

$$=\frac{1}{4}[\|4(_AS(f,f))\|+i\|2(_AS(f,f))\|-i\|2_AS(f,f)\|]=\|_AS(f,f)\|=p_A^2(f,f)$$

Proposition 6. *Let p_A be a semi-quadratic A-form on V and $_AS$ be a left sesquilinear A-form. Then for all $f,g,h\in V$:*

$$p_A^2(f+g)+p_A^2(f+h)\leq (p_A^2(f)+p_A^2(g))^2+(p_A^2(f)+p_A^2(h))^2$$

Proof.

$$p_A^2(f+g)+p_A^2(f+h)=\|_AS(f+g,f+g)\|+\|_AS(f+h,f+h)\|$$

$$=\|_AS(f,f)+_AS(f,g)+_AS(g,f)+_AS(g,g)\|$$

$$+\|_AS(f,f)+_AS(f,h)+_AS(h,f)+_AS(h,h)\|$$

$$=\|_AS(f,f)+_AS(f,g)+(_AS(f,g))^*+_AS(g,g)\|$$

$$+\|_AS(f,f)+_AS(f,h)+(_AS(f,h))^*+_AS(h,h)\|$$

$$\leq\|_AS(f,f)\|+2Re\|_AS(f,g)\|+\|_AS(g,g)\|$$

$$+\|_AS(f,f)\|+2Re\|_AS(f,h)\|+\|_AS(h,h)\|$$

$$=p_A^2(f)+2q_A(f)q_A(g)+p_A^2(g)+p_A^2(f)+2q_A(f)q_A(h)+p_A^2(h)$$

$$=(p_A(f)+p_A(g))^2+(p_A(f)+p_A(h))^2$$

Theorem 5. *Let H be a complex vector space, $_AS$ is a left sesquilinear A-form on H, and q_A the quadratic A- form generated by S_A. Then for all $f,g\in H$:*

$$_AS(f,g)=\frac{1}{4}\{q_A(f+g)-q_A(f-g)+iq_A(f+ig)-iq_A(f-ig)\}.$$

Proof.

Expanding each term gives:

$$q_A(f+g) =_A S(f+g, f+g) =_A S(f,f) +_A S(f,g) +_A S(g,f) +_A S(g,g) \quad (1)$$

$$-q_A(f-g) = -_A S(f-g, f-g) = -_A S(f,f) +_A S(f,g) +_A S(g,f) -_A S(g,g) \quad (2)$$

$$q_A(f+ig) =_A S(f+ig, f+ig) =_A S(f,f) +_A S(f,ig) +_A S(ig,f) +_A S(ig,ig)$$

$$=_A S(f,f) +_A S(f,g)i^* + i({}_AS(g,f)) + i({}_AS(g,g))i^*$$

$$iq_A(f+ig) = i({}_AS(f,f)) + i({}_AS(f,g))i^* + i^2({}_AS(g,f)) + i^2({}_AS(g,g))i^*$$

$$= i({}_AS(f,f)) + i({}_AS(f,g))i^* -_A S(g,f) -_A S(g,g)i^*$$

$$iq_A(f+ig) = i({}_AS(f,f)) + i({}_AS(f,g))i^* -_A S(g,f) -_A S(g,g)i^* \quad (3)$$

$$q_A(f-ig) =_A S(f-ig, f-ig) =_A S(f,f) +_A S(f,-ig) +_A S(-ig,f) +_A S(-ig,-ig)$$

$$=_A S(f,f) +_A S(f,g)(-i)^* - i({}_AS(g,f)) - i({}_AS(g,g))(-i)^*$$

$$=_A S(f,f) -_A S(f,g)i^* - i({}_AS(g,f)) + i({}_AS(g,g))i^*$$

$$-iq_A(f-ig) = -i({}_AS(f,f)) + i({}_AS(f,g))i^* -_A S(g,f) +_A S(g,g)i^* \quad (4)$$

Adding these four expressions and dividing by 4 yields

$$\text{R.H.S} = \tfrac{1}{4}\{q_A(f+g) - q_A(f-g) + iq_A(f+ig) - iq_A(f-ig)\}$$

$$= \tfrac{1}{4}\{2({}_AS(f,g)) + 2({}_AS(g,f)) + 2i({}_AS(f,g))i^* - 2({}_AS(g,f))\} =_A S(f,g)$$

which proves the polarization identity.

Theorem 6. *Let $_AS$ be a left sesquilinear A-form on a vector space H, and let q_A be the corresponding quadratic A-form on H.Then for all $\alpha, \beta \in H$:*

$$q_A(\alpha+\beta)+q_A(\alpha-\beta)=2[q_A(\alpha)+q_A(\beta)]$$

Proof.

$$q_A(\alpha+\beta)+q_A(\alpha-\beta) =_A S(\alpha+\beta,\alpha+\beta)+_A S(\alpha-\beta,\alpha-\beta)$$

$$=_A S(\alpha,\alpha)+_A S(\alpha,\beta)+_A S(\beta,\alpha)+_A S(\beta,\beta)$$

$$+_A S(\alpha,\alpha)+_A S(\alpha,-\beta)+_A S(-\beta,\alpha)+_A S(-\beta,-\beta)$$

$$=_A S(\alpha,\alpha)+_A S(\alpha,\beta)+_A S(\beta,\alpha)+_A S(\beta,\beta)$$

$$+_A S(\alpha,\alpha)-_A S(\alpha,\beta)-_A S(\beta,\alpha)+_A S(\beta,\beta)$$

$$=2(_A S(\alpha,\alpha))+2(_A S(\beta,\beta))=2[q_A(\alpha)+q_A(\beta)]$$

Definition 14. *Let X be an A-vector space over $\mathbb{C}$. An A-normed space on X is a map $\|.\|_A : X \to A$ satisfying the following conditions for all $x, y \in X$ and $\lambda \in \mathbb{C}$*

(1) $\|x\|_A = 0_A \Longleftrightarrow x = 0_A$.

(2) $\|\lambda x\|_A = \|\lambda\|_A \|x\|_A$,

(3) $\|x+y\|_A \leq \|x\|_A + \|y\|_A$.

A-normed space is a pair $(X, \|.\|_A)$ where X is an A-vector space and $\|.\|_A$ is an A-norm on X.

Example 10. *Let X be an A-vector space , define $\|.\|_A : X \to A$*

$$\|x\|_A = |x| I_A$$

Then $(X, \|.\|_A)$ is A-norm space.

Proof.

(1) $|x|I_A \geq 0_A \Longrightarrow \|x\|_A \geq 0_A$.

(2) $\|\lambda x\|_A = |\lambda x| I_A = |\lambda||x| I_A = |\lambda| \|x\|_A$.

(3) $\|x+y\|_A = |x+y| I_A \leq (|x|+|y|) I_A = |x| I_A + |y| I_A = \|x\|_A + \|y\|_A$

Hence, the axioms hold and $(X, \|.\|_A)$ is A-normed space.

Definition 15. *Let X, Y be A-normed spaces. A linear operator $T_A : X \to Y$ is called A-bounded if there exists $c_A \in A$ such that for all $x \in X$*

$$\|T_A x\|_A \leq c_A \|x\|_A.$$

Lemma 1. *Let T_A be a sesquilinear A-form operator, then the operator norm of T_A given by*

$$\|T\|_A = \sup_{x\in X} \frac{\|T_A x\|}{\|x\|} I_A = \sup_{\|x\|=1_A} \|T_A x\| I_A$$

defines an A-norm.

Proof.

(1) $\sup_{x\in X,\|x\|=1_A} \frac{\|T_A x\|}{\|x\|} I_A \geq 0_A \implies \|T_A\|_A \geq 0_A$, If $\|T_A\|_A = 0_A \implies \sup_{x\in X,\|x\|=1_A} \frac{\|T_A x\|}{\|x\|} I_A = 0_A \implies \sup |T_A x\| I_A = 0_A \implies T_A = 0_A$.

(2) $\|\lambda T_A x\|_A = \sup_{x\in X,\|x\|=1_A} \frac{\|\lambda T_A x\|}{\|x\|} I_A = \|\lambda\| \sup_{x\in X,\|x\|=1_A} \frac{\|T_A x\|}{\|x\|} I_A = \|\lambda\| \|T_A x\|_A$.

(3) Now we will prove that $\|(T_1 + T_2)x\|_A \leq |T_1\|_A + \|T_2\|_A$

$$\begin{aligned}\|(T_1 + T_2)x\|_A &= \sup_{\|x\|=1_A} \|(T_1 + T_2)x\| I_A \\ &= \sup_{\|x\|=1_A} \|T_1 x + T_2 x\| I_A \\ &\leq \sup_{\|x\|=1_A} \|T_1 x\| I_A + \sup_{\|x\|=1_A} \|T_2 x\| I_A \\ &= \|T_1\|_A + \|T_2\|_A.\end{aligned}$$

Example 11. *The identity operator $T_A : A \to A$ on A-normed space is bounded and A-norm of $\|T_A\| = 1_A$*

Example 12. *Let the matrix $A = (\alpha_{ik})_{r\times n} \cdot I_A$ define on operator $T_A : A^n \to A^r$ by*

$$T_A(x) = Ax$$

where $x = (\xi_i)$ and $y = (\delta_i)$ are column vectors with n and r components respectively. Then T_A is linear and bounded.

Proof. We first prove that T_A is linear

$$T_A(\lambda x) = \sum_{i=1}^{r}\sum_{k=1}^{n}\alpha_{ik}\cdot I_A(\lambda\xi_k)$$

$$= \lambda(\sum_{i=1}^{r}\sum_{k=1}^{n}\alpha_{ik}\cdot I_A(\xi_k)) = \lambda T_A(x)$$

$$T_A(x+y) = \sum_{i=1}^{r}\sum_{k=1}^{n}\alpha_{ik}\cdot I_A(\xi+\delta)_k$$

$$= \sum_{i=1}^{r}\sum_{k=1}^{n}\alpha_{ik}\cdot I_A(\xi_k+\delta_k)$$

$$= \sum_{i=1}^{r}\sum_{k=1}^{n}[\alpha_{ik}\cdot I_A\xi_k + \alpha_{ik}\cdot I_A\delta_i]$$

$$= \sum_{i=1}^{r}\sum_{k=1}^{n}\alpha_{ik}\cdot I_A\xi_k + \sum_{i=1}^{r}\sum_{k=1}^{n}\alpha_{ik}\cdot I_A\delta_k$$

$$= T_A(x) + T_A(y)$$

Now let us prove that T_A is bounded:

$$\|T_Ax\|^2 = \sum_{i=1}^{r}[\sum_{k=1}^{n}\alpha_{ik}\cdot I_A(\xi_k)]^2$$

$$\leq \sum_{i=1}^{r}[(\sum_{k=1}^{n}\alpha_{ik}^2\cdot I_A)^{\frac{1}{2}}(\sum_{k=1}^{n}\xi_k^2)^{\frac{1}{2}}]^2$$

$$= [\sum_{i=1}^{r}\sum_{k=1}^{n}\alpha_{ik}^2\cdot I_A]\|x\|^2$$

$\Rightarrow \|T_Ax\|^2 \leq c_A^2\|x\|^2$,then $\|T_Ax\| \leq c_A\|x\|$ where $c_A = \sum_{i=1}^{r}\sum_{k=1}^{n}\alpha_{ik}\cdot I_A$.

Definition 16. *Let X, Y are A-normed space and $a \in A$ such that for all $x, y \in A$*

$$\|T_A(x,y)\| \leq \|x\|\|y\|I_A$$

Then T_A is said to be bounded, and the expression

$$\|S\| = \sup_{x\in X, y\in Y}[\frac{\|T_A(x,y)\|I_A}{\|x\|\|y\|}] = \sup_{\|x\|=1_A,\|y\|=1_A}\|T_A(x,y)\|I_A$$

is called the A-norm of S.

Theorem 7. *Let H_1, H_2 be Hilbert spaces and $T_A : H_1 \times H_2 \to A$ be a bounded sesquilinear A-form, then T_A has a representation of the form*

$$T_A(x,y) = (Sx, y)$$

where $S : H_1 \to H_2$ is a bounded linear A-form and its norm is $\|S\| = \|T_A\|$.

Proof. Let y be a variable and keep x fixed.

Let $\overline{T_A(x,y)} = (y,z) \Rightarrow T_A(x,y) = (z,y)$ where $z \in H_2, x \in H_1$

Let $S : H_1 \longrightarrow H_2$ be defined as

$$Sx = z$$

$\Rightarrow T_A(x, y) = (Sx, y)$

Now we will prove that S is linear

$$(S(\alpha x_1, \beta x_2), y) = T_A(\alpha x_1, \beta x_2, y)$$

$$= T_A(\alpha x_1, y) + T_A(\beta x_2, y)$$

$$= \alpha T_A(x_1, y) + \beta T_A(x_2, y)$$

$$= \alpha(Sx_1, y) + \beta(Sx_2, y)$$

$$= (\alpha Sx_1 + \beta Sx_2, y)$$

$\Rightarrow S(\alpha x_1, \beta x_2) = \alpha Sx_1 + \beta Sx_2$. So S is linear

$$\|T\| = \sup_{\|x\| \neq 0_A, \|y\| \neq 0_A}[\tfrac{\|T_A(x,y)\| I_A}{\|x\|\|y\|}]$$

$$= \sup_{x \neq 0_A, Sx \neq 0_A}[\tfrac{\|(Sx,y)\| I_A}{\|x\|\|y\|}]$$

$$\geq \sup_{x \neq 0_A, Sx \neq 0_A}[\tfrac{\|(Sx,Sx)\| I_A}{\|x\|\|Sx\|}]$$

$$= \sup_{x \neq 0_A, Sx \neq 0_A}[\tfrac{\|Sx\| I_A}{\|x\|}] = \|S\|$$

$\Rightarrow \|T_A\| \geq \|S\| \longrightarrow (boundedness)$

$$\|T_A\| = \sup_{x \neq 0_A, Sx \neq 0_A}[\tfrac{\|(Sx,y)\| I_A}{\|x\|\|y\|}]$$

$$\leq \sup_{x \neq 0_A, y \neq 0_A}[\tfrac{\|Sx\|\|y\| I_A}{\|x\|\|y\|}] = \|S\|$$

$\Rightarrow \|T_A\| = \|S\|$.

4. Conclusion

In this paper, we introduced the definition of sesquilinear on C^*-algebra, which we called the sesquilinear A-form, and we also presented the definitions of right and left sesquilinear A-forms, bilinear A-form, and provided some examples to illustrate our definitions. Finally, we established some important properties of the sesquilinear A-form.

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