Journal of Contemporary Applied Mathematics
V. 16, No 1, 2026, January-June
ISSN 2222-5498, E-ISSN 3006-3183
https://doi.org/10.62476/jcam.161.6

On basicity of some trigonometric system in Banach Function Spaces

B.T. Bilalov*, V.F. Salmanov, T. Hagverdi

Abstract. In this paper it is considered the trigonometric system $\{1; \cos nx; x \sin nx\}_{n\in\mathbb{N}}$, which is a collection of eigenfunctions of one nonlocal spectral problem for an ordinary second order differential operator. Let $X(-\pi,\pi)$ be a Banach Function Space (by Luxembourg classification) on $(-\pi,\pi)$ with Lebesgue measure. A criterion is obtained for the trigonometric system $\left\{\frac{1}{2}; \cos nt; \sin nt\right\}_{n\in\mathbb{N}}$ to have the Riesz Property in $X(-\pi,\pi)$. It is proved that if the trigonometric system has the Riesz Property in $X(-\pi,\pi)$, then the system $(\mathscr{T})$ also forms a basis for $X(-\pi,\pi)$. Some concrete spaces, such as the weighted Lebesgue space $L_{p;w}(-\pi,\pi)$, the weighted grand Lebesgue space $L_{p);w}(-\pi,\pi)$, Lebesgue space with variable exponent $L_{p(\cdot)}(-\pi,\pi)$, Morrey space $L_{p;\lambda}(-\pi,\pi)$, symmetric space $X(-\pi,\pi)$ with Boyd indices $\alpha_X; \beta_X \in (0,1)$ are presented.
Key Words and Phrases: Trigonometric system, Banach Function Space, nonlocal problem, basicity, Riesz property.
2010 Mathematics Subject Classifications: 33B10, 46E30, 54D70

1. Introduction

Consider the following trigonometric system

$$(\mathscr{T}) = \left\{u_n^+(x); u_{n+1}^-(x)\right\}_{n\in\mathbb{Z}_+},$$

where

$$u_n^+(x) \equiv \cos nx;\; u_{n+1}^-(x) = x\sin(n+1)x,\; n\in\mathbb{Z}_+.$$

This system arises when we solve the following nonlocal boundary value problem for a degenerate elliptic equation

$$\left.\begin{array}{l} y^m u_{xx} + u_{yy} = 0,\; 0 < x < 2\pi,\; y > 0, \\ u(x,0) = f(x);\; u(0,y) = u(2\pi;y), \\ u_x(0;y) = 0,\; y > 0, \end{array}\right\} \qquad (1)$$

*Corresponding author.

is considered by some authors (see e.g. [1, 2, 3, 4, 5, 6, 7, 8]), where $m > -2$ is some number. Solving this problem by the method of separation of variables leads to the spectral problem with respect to the variable x:

$$\left.\begin{array}{l}\Phi''(x) + \lambda^2 \Phi(x) = 0,\ 0 < x < 2\pi, \\ \Phi(0) = \Phi(2\pi);\ \Phi'(0) = 0\,.\end{array}\right\} \tag{2}$$

It is not hard to see that $(\mathscr{T})$ is the system of all eigenfunctions of the spectral problem (2). Note that the problem (1) has some features compared to the general theory of elliptic equations. Therefore, when solving the problem (1) we need to investigate the basis properties (completeness, minimality, basicity) of the system $(\mathscr{T})$ in the corresponding Banach Function Space (BFS), associated with the considered Banach-Sobolev space.

In this paper it is considered the trigonometric system $\{1; \cos nx; x \sin nx\}_{n\in\mathbb{N}}$, which is a collection of eigenfunctions of one nonlocal spectral problem for an ordinary second order differential operator. Let $X(-\pi,\pi)$ be a Banach Function Space (by Luxembourg classification) on $(-\pi,\pi)$ with Lebesgue measure. A criterion is obtained for the trigonometric system $\left\{\frac{1}{2}; \cos nt; \sin nt\right\}_{n\in\mathbb{N}}$ to have the Riesz Property in $X(-\pi,\pi)$. It is proved that if the trigonometric system has the Riesz Property in $X(-\pi,\pi)$, then the system $(\mathscr{T})$ also forms a basis for $X(-\pi,\pi)$. Some concrete spaces, such as the weighted Lebesgue space $L_{p;w}(-\pi,\pi)$, the weighted grand Lebesgue space $L_{p);w}(-\pi,\pi)$, Lebesgue space with variable exponent $L_{p(\cdot)}(-\pi,\pi)$, Morrey space $L_{p;\lambda}(-\pi,\pi)$, symmetric space $X(-\pi,\pi)$ with Boyd indices $\alpha_X; \beta_X \in (0,1)$ are presented.

2. Preliminaries

First, let us assume some notation. $\mathbb{N}$ be the set of natural numbers, $\mathbb{Z}_+ = \{0\} \cup \mathbb{N}$; $J = (0, 2\pi)$. $\mathbb{R}$ be the set of real numbers: $\mathbb{C}$ be the set of complex numbers; $L[M]$ is a linear span of a set M. $\bar{M}$ is a closure of the set $M \subset \mathbb{R}$; $|M|$ is a Lebesgue measure of the set $M \subset \mathbb{R}$. By $\mathscr{L}_0(J)$ we denote all measurable (in Lebesgue sense) functions on J. Assume that $\mathscr{L}_0(J)$ is a linear space over a field K ($K \equiv \mathbb{R}$ or $K \equiv \mathbb{C}$). $C_0^\infty(J)$ is the set of all infinitely differentiable functions on J with compact support in J.

So, let us recall the definition of BFS.

Definition 1. *[9]* $\|\cdot\|_{X(J)} : \mathscr{L}_0(J) \to \bar{R}_+ = [0, +\infty]$ *is called a Banach Function norm, iff:*

i) $\|\cdot\|_{X(J)}$ *is a norm on* $\mathscr{L}_0(J)$*;*

ii) $f; g \in \mathscr{L}_0(J) : |f| \le |g|$ *a.e. on* $J \Rightarrow \|f\|_{X(J)} \le \|g\|_{X(J)}$*;*

iii) Fatou property. $|f_n| \uparrow |f|$, $n \to \infty \Rightarrow \|f_n\|_{X(J)} \uparrow \|f\|_{X(J)}$*;*

iv) $\forall E \subset J$ *(measurable in Lebesgue sense)*$\Rightarrow \|\chi_E\|_{X(J)} < +\infty$*;*

v) it holds the continuous embedding $X(J) \subset L_1(J)$*,*

where $L_p(J)$, $1 \le p < +\infty$, *denotes the ordinary Lebesgue space with the norm*

$$\|f\|_{L_p(J)} = \left(\int_J |f|^p \, dt \right)^{\frac{1}{p}}.$$

BFS $X(J)$ is defined by

$$X(J) = \left\{ f \in \mathscr{L}_0(J) : \|f\|_{X(J)} < +\infty \right\}.$$

$X(J)$ with the norm $\|\cdot\|_{X(J)}$ is a Banach space. By $X'(J)$ denote the associate space of $X(J)$, which is defined by the following Banach Function norm

$$\|g\|_{X'(J)} = \sup_{\|f\|_{X(J)} \le 1} \int_J |fg| \, dx.$$

Denote by $\mu_f(\cdot)$ a distribution function of $f \in \mathscr{L}_0(J)$, i.e.

$$\mu_f(\lambda) = |\{x \in J : |f(x)| > \lambda\}|, \; \forall \lambda \ge 0.$$

If $f; g \in X(J)$: $\mu_f(\lambda) = \mu_g(\lambda)$, $\forall \lambda > 0, \Rightarrow \|f\|_{X(J)} = \|g\|_{X(J)}$, then $X(J)$ is called a symmetric space.

More information about BFS one can get from the monographs [9, 10, 11].

We will also need some facts from the work [12]. First, we define the Muckenhoupt class $A_p(J)$ of weight functions $w : J \to \bar{R}_+$, which is assumed to be 2π-periodically extended to the entire real axis R. We say that $w \in A_p(J)$, $1 < p < +\infty$, iff

$$\sup_{I \subset R} \left(\frac{1}{|I|} \int_I w(x) \, dx \right) \left(\frac{1}{|I|} \int_I [w(x)]^{-\frac{1}{p-1}} \, dx \right)^{p-1} < +\infty,$$

where sup is taken over any interval $I \subset R$.

Consider the following classical trigonometric system

$$\left\{ \frac{1}{2}; \cos nx; \sin nx \right\}_{n \in N}. \tag{3}$$

Let us make some comments regarding the basicity of the system (3) in the weighted Lebesgue space $L_{p;w}(J)$ with the norm

$$\|f\|_{L_{p;w}(J)} = \left(\int_J |f|^p \, w dx \right)^{\frac{1}{p}}.$$

For a function $f \in L_{p;w}(J)$ we denote by $\left\{f_n^c; f_{n+1}^s\right\}_{n\in\mathbb{Z}_+}$ the Fourier coefficients of the function f on the trigonometric system (3):

$$\vartheta_n^c(f) = f_n^c = \frac{1}{\pi}\int_0^{2\pi} f(t)\cos nt\,dt\ ,\quad \forall n \in \mathbb{Z}_+;$$

$$\vartheta_n^s(f) = f_n^s = \frac{1}{\pi}\int_0^{2\pi} f(t)\sin nt\,dt\ ,\quad \forall n \in \mathbb{N}. \tag{4}$$

Let

$$S_n(f)(t) = \frac{1}{2}f_0^c + \sum_{k=1}^{n}\left(f_k^c\cos kt + f_k^s\sin kt\right)\ ,\ n \in \mathbb{N},$$

be the partial sums of the Fourier sums of f on system (3). The following theorem on the basicity of the system (3) in $L_{p;w}(J)$ immediately follows from the results of the work [12] .

Theorem 1. *[12] The system* (3) *forms a basis in* $L_{p;w}(J)$*,* $1 < p < +\infty$*, if and only if* $w \in A_p(J)$*.*

In fact, the basicity of the system (3) in $L_{p;w}(J)$ is equivalent to the basicity of the same system in $L_{p;w_0}(-\pi,\pi)$, where $w_0(t) = w(t+\pi)$, $t \in (-\pi,\pi)$. And it is evident that $w \in A_p(J) \Leftrightarrow w_0 \in A_p(-\pi,\pi)$. Further, it directly follows from the Theorem 8 in [12]. Note that basicity of the system (3) is understood in the following sense

$$\|S_n(f) - f\|_{L_{p;w}(J)} \to 0,\ n \to \infty.$$

The following question arises: let the system (3) form a basis for $L_{p;w_0}(-\pi,\pi)$, $1 < p < +\infty$. For $\forall f \in L_{p;w_0}(-\pi,\pi)$ when the series

$$S^c(f) = \frac{1}{2}f_0^c + \sum_{k=1}^{\infty} f_k^c\cos kt,$$

and

$$S^s(f) = \sum_{k=1}^{\infty} f_k^s\sin kt, \tag{5}$$

converge in $L_{p;w_0}(-\pi,\pi)$?

For this purpose, we first define BFS $X(-\pi,\pi)$ on $(-\pi,\pi)$ by the relation

$$X(-\pi,\pi) = \left\{f \in \mathscr{L}_0(-\pi,\pi):\ f(\cdot - \pi) \in X(J)\right\},$$

with the norm

$$\|f\|_{X(-\pi,\pi)} = \|f(\cdot-\pi)\|_{X(J)}\,.$$

Assume the following

Definition 2. *We will call BFS $X(J)$ a (R)-space, if $f(\cdot) \in X(J) \Rightarrow f(2\pi - \cdot) \in X(J)$.*

It is not difficult to check that (R)-space property of $X(J)$ in term of BFS $X(-\pi,\pi)$ is equivalent to the condition

$$f(\cdot) \in X(-\pi,\pi) \Leftrightarrow f(-\cdot) \in X(-\pi,\pi). \tag{6}$$

That's why, we will hold further discussion regarding the BFS $X(-\pi,\pi)$.

Let the system (3) form a basis for $X(-\pi,\pi)$. Denote

$$\overline{L\left[\{\cos nt\}_{n\in\mathbb{Z}_+}\right]} \equiv X^c(-\pi,\pi); \overline{L\left[\{\sin nt\}_{n\in\mathbb{N}}\right]} \equiv X^s(-\pi,\pi),$$

where the closure is taken under the norm of $X(-\pi,\pi)$. Let $X(-\pi,\pi)$ be a (R)-space. Take $\forall f \in X(-\pi,\pi)$ and represent it in the form

$$f(t) = f^+(t) + f^-(t),$$

where $f^+(t) = \frac{f(t)+f(-t)}{2}$ is an even, $f^-(t) = \frac{f(t)-f(-t)}{2}$ is an odd part of $f(\cdot)$. From (6) follows that $f^\pm \in X(-\pi,\pi)$. Expand the function $f(\cdot)$ on basis (3):

$$f(t) = \frac{1}{2}f_0^c + \sum_{k=1}^{\infty}(f_k^c \cos kt + f_k^s \sin kt),$$

so that

$$\|S_n(f) - f(\cdot)\|_{X(-\pi,\pi)} \to 0,\ n\to\infty.$$

Consider

$$S_n^c(f) = \frac{1}{2}f_0^c + \sum_{k=1}^{n} f_k^c \cos kt = \frac{1}{2}\vartheta_0^c(f) + \sum_{k=1}^{n}\vartheta_k^c(f)\cos kt =$$

$$= \vartheta_0^c\left(f^+\right) + \sum_{k=1}^{n}\vartheta_k^c\left(f^+\right)\cos kt + \vartheta_0^c\left(f^-\right) + \sum_{k=1}^{n}\vartheta_k^c\left(f^-\right)\cos kt =$$

/ due to the oddness of $f^-(\cdot)$/ $= \vartheta_0^c(f^+) + \sum_{k=1}^{n}\vartheta_k^c(f^+)\cos kt =$

/ due to the evenness of $f^+(\cdot)$/

$$= \vartheta_0^c\left(f^+\right) + \sum_{k=1}^{n}\left(\vartheta_k^c\left(f^+\right)\cos kt + \vartheta_k^s\left(f^+\right)\sin kt\right) = S_n\left(f^+\right),\ \forall n \in \mathbb{Z}_+.$$

From here it directly follows that the sequence $\{S_n^c(f)\}_{n\in\mathbb{Z}_+}$ converges for $\forall f \in X(-\pi,\pi)$ in $X(-\pi,\pi)$ and that's why it is clear that the sequence $\{S_n^s(f)\}_{n\in\mathbb{N}}$ also converges, where

$$S_n^s(f) = \sum_{k=1}^{n} \vartheta_k^s(f)\sin kt,\ \ n\in\mathbb{N}.$$

As a result we obtain

$$f = S(f) = S^c(f) + S^s(f)\,,\ \ \forall f\in X(-\pi,\pi)\,, \tag{7}$$

where

$$S^s(f) = \sum_{k=1}^{\infty} \vartheta_k^s(f)\sin kt.$$

It is evident that $S^c(f)\in X^c(-\pi,\pi)\,\&S^s(f)\in X^s(-\pi,\pi)$. Moreover, it is obvious that $X^c(-\pi,\pi)\bigcap X^s(-\pi,\pi) = \{0\}$. Then from (7) follows that the subspaces $X^c(-\pi,\pi)$ and $X^s(-\pi,\pi)$ are complemented in $X(-\pi,\pi)$, moreover it is valid the direct sum

$$X(-\pi,\pi) = X^c(-\pi,\pi)\dot{+}X^s(-\pi,\pi)\,. \tag{8}$$

Also from here it follows that regarding the system (3) in BFS $X(-\pi,\pi)$ it is true so called the Riesz Property, which we define as follows:

Definition 3. *We will say that the trigonometric basis* (3) *of BFS* $X(-\pi,\pi)$ *has the Riesz property in* $X(-\pi,\pi)$, *if* $\exists C>0$: *for* $\forall f\in X(-\pi,\pi)$ *it holds*

$$\left.\begin{array}{l} \|S_n^c(f)\|_{X(-\pi,\pi)} \le C\,\|f\|\,,\ \forall n\in\mathbb{Z}_+, \\ \|S_n^s(f)\|_{X(-\pi,\pi)} \le C\,\|f\|\,,\ \forall n\in\mathbb{N}. \end{array}\right\} \tag{9}$$

As a result, we obtain that if $X(J)$ is a (R)-space, then the system (3) has the Riesz Property.

Let us prove the converse, namely, let the system (3) form a basis for $X(-\pi,\pi)$ and it has the Riesz Property. Prove that then $X(-\pi,\pi)$ is a (R)-space. Assume $X(-\pi,\pi)$ is not a (R)-space. Then there exists a function $f\in X(-\pi,\pi)$ such that $f(-x)\notin X(-\pi,\pi)$. Hence it follows that the series

$$S(f(-x))(t) = \frac{1}{2}\vartheta_0^c(f(-x)) + \sum_{k=1}^{\infty}\left(\vartheta_k^c(f(-x))\cos kt + \vartheta_k^s(f(-x))\sin kt\right),$$

does not converge in $X(-\pi,\pi)$. Since the system (3) has the Riesz Property , it is evident that the series $S^c(f)$ converges in $X(-\pi,\pi)$. Then from the relation

$$S_n^c(f) = S_n\left(f^+\right)\,,\ \forall n\in\mathbb{Z}_+,$$

it follows that the series $S(f^+)$ converges in $X(-\pi,\pi)$ and as a result from the expression $S(f^+) = \frac{1}{2}S(f) + \frac{1}{2}S(f(-x))$ follows that the series $S(f(-x))$ also converges in $X(-\pi,\pi)$. Then from the basicity of the system (3) in $X(-\pi,\pi)$ it follows that $f(-x) \in X(-\pi,\pi)$. We have arrived at a contradiction. Therefore, it is valid the following

Proposition 1. *Let the system* (3) *form a basis for BFS $X(-\pi,\pi)$. Then this system has the Riesz Property if and only if $X(-\pi,\pi)$ is a (R)-space.*

3. Main results

In this section we present one method for establishing the basicity of the system $(\mathscr{T})$ in BFS $X(J)$. To do this, we use the Riesz Property of the basis (3) in $X(-\pi,\pi)$. Thus, it is valid the following main

Theorem 2. *Let BFS $X(J)$ be a (R)-space, in which the set $C_0^\infty(J)$ is dense and the trigonometric system* (3) *form a basis for it. Then the system $(\mathscr{T})$ also forms a basis for $X(J)$ and it has the Riesz Property, i.e. $\exists C > 0$, $\forall f \in X(J)$ it holds*

$$\left\| \sum_{k=0}^{n} e_k^+(f) \cos kx \right\|_{X(J)} \leq C \|f\|_{X(J)}, \ \forall n \in \mathbb{Z}_+,$$

$$\left\| \sum_{k=1}^{n} e_k^-(f)\ x \sin kx \right\|_{X(J)} \leq C \|f\|_{X(J)}, \ \forall n \in \mathbb{N},$$

where $\left\{e_0^+; e_k^+; e_k^-\right\}_{k\in\mathbb{N}} \subset X^*(J)$ is a system biorthogonal to the basis $(\mathscr{T})$.

We will prove this theorem using the basicity criterion in Banach spaces, i.e. first we will establish the minimality and completeness of the system $(\mathscr{T})$ in $X(J)$ and then the uniformly boundedness of corresponding projectors.

3.1. Minimality

Consider the following system of functions

$$\varphi_0(x) = \frac{2\pi - x}{2\pi^2}; \varphi_k(x) = \frac{2\pi - x}{\pi^2} \cos kx; \psi_k(x) = \frac{1}{\pi^2} \sin kx, \quad k \in \mathbb{N},$$

and define the functionals

$$e_k^+(f) = \int_0^{2\pi} f(x)\, \varphi_k(x)\, dx, \ \forall k \in \mathbb{Z}_+;$$

$$e_k^-(f) = \int_0^{2\pi} f(x)\,\psi_k(x)\,dx\,,\ \forall k \in \mathbb{N}.$$

It is true the following

Lemma 1. *The system* $(\mathscr{T})$ *is minimal in BFS* $X(J)$.

Proof. Directly by simple calculations one can verify the validity of the following relations

$$\left.\begin{array}{ll} e_k^+(u_n^+) = \delta_{kn}, & e_k^+\left(u_j^-\right) = 0; \\ e_i^-(u_k^+) = 0, & e_i^-\left(u_j^-\right) = \delta_{ij}. \end{array}\right\} \qquad (10)$$

Paying attention to the fact that the functions $\{\varphi_k; \psi_i\}$ are uniformly bounded, from continuous embedding $X(J) \subset L_1(J)$ we immediately obtain

$$\left|e_k^\pm(f)\right| \le c \int_0^{2\pi} |f(t)|\,dt \le c\,\|f\|_{X(J)}\,,$$

for possible values of index k, where $c > 0$ is some constant independent of f. From here it directly follows $\left\{e_k^+; e_i^-\right\} \subset X^*(J)$. Then the minimality of system $\left\{u_i^+; u_j^-\right\}$ in $X(J)$ follows from the relation (10).

Lemma is proved.

3.2. Completeness

Let us establish the completeness of the system $(\mathscr{T})$ in $X(J)$. So, it is valid

Lemma 2. *Let the set* $C_0^\infty(J)$ *be dense in BFS* $X(J)$. *Then the system* $(\mathscr{T})$ *is complete in* $X(J)$.

Proof. Since $C_0^\infty(J)$ is dense in $X(J)$, it is sufficient to prove that any function $f \in C_0^\infty(J)$ can be approximated by linear combination of the system $(\mathscr{T})$. Take $\forall f \in C_0^\infty(J)$ and assume $g(x) = \frac{2\pi - x}{\pi^2} f(x)$. It is evident that $g \in C_0^\infty(J)$. Consider

$$f_n^+ = \frac{1}{\pi^2}\int_0^{2\pi} f(x)(2\pi - x)\cos nx dx = \int_0^{2\pi} g(x)\cos nx dx\,,\ n \in \mathbb{N}.$$

Integrating by parts twice we have

$$f_n^+ = -\frac{1}{n}\int_0^{2\pi} g'(x)\sin nx dx = \frac{1}{n^2}\int_0^{2\pi} g''(x)\cos nx dx \ \Rightarrow$$

$$\Rightarrow \left|f_n^+\right| \leq \frac{c}{n^2}\,,\ \forall n \in \mathbb{Z}_+,$$

where $c > 0$ is a constant independent of n. Similarly, we get

$$\left|f_n^-\right| \leq \frac{c}{n^2}\,,\ \forall n \in \mathbb{N}.$$

As a result, it is established that the series

$$F(x) = f_0^+ + \sum_{n=1}^{\infty} \left(f_n^+ \cos nx + f_n^- x \sin nx\right), \tag{11}$$

uniformly converges to a function $F \in C\left(\bar{J}\right)$ on J. Due to the fact that the system $(\mathscr{T})$ forms a basis (see, [5]) for $L_2(J)$, it holds $F(x) \equiv f(x)$. Then it is not hard to see that the series (11) also converges to f in $X(J)$. Hence, it immediately follows the completeness of the system $(\mathscr{T})$ in $X(J)$.

Lemma is proved.

3.3. Proof of Theorem 2

Paying attention to the Lemmas 1 & 2, it is sufficient to prove the uniformly boundedness of the following projectors

$$S_{n;m}(f) = \sum_{k=0}^{n} e_k^+(f) \cos kx + \sum_{k=1}^{m} e_k^-(f)\, x \sin kx\,,\ \forall (n;m) \in \mathbb{Z}_+ \times \mathbb{N}.$$

We have

$$e_0^+(f) = \frac{1}{2\pi^2} \vartheta_0^c(g)\,, e_k^+(f) = \frac{1}{\pi^2} \vartheta_k^c(g)\,,\ \forall k \in \mathbb{N},$$

($\vartheta_k^c(g)$ are the Fourier coefficients of g), where $g(x) = (2\pi - x) f(x)\ , x \in J$.

Taking into account these relations and the condition that $X(-\pi,\pi)$ is a (R)-space, using Proposition 1, we get

$$\|S_{n;m}(f)\|_{X(J)} \leq \left\|\frac{1}{2\pi^2} \vartheta_0^c(g)\right\|_{X(J)} + \frac{1}{\pi^2} \left\|\sum_{k=1}^{n} \vartheta_k^c(g) \cos kx\right\|_{X(J)} + 4 \left\|x \sum_{k=1}^{m} \vartheta_k^s(f)\right\|_{X(J)} \leq$$

$$\leq c\,\|f\|_{X(J)}\,,\ \forall (n;m) \in \mathbb{Z}_+ \times \mathbb{N}, \tag{12}$$

where the constant $c > 0$ is independent of $(n;m)$ and f.

It directly follows from the inequality (12) that the system $(\mathscr{T})$ has the Riesz property in$X(J)$.

Theorem is proved.

4. Concrete BFS

4.1. The weighted Lebesgue space $L_{p;w}(J)$

Let $w \in A_p(J)\,,\, 1 < p < +\infty$, and $w(2\pi - x) = w(x)$ hold, a.e. on J. Assume $w_0(t) = w(t+\pi)$, $\forall t \in (-\pi, \pi)$. For $w_0(\cdot)$ we obtain $w_0(-t) = w(-t+\pi) = w(2\pi - (-t+\pi)) = w(\pi + t) = w_0(t)$, $\forall t \in (-\pi, \pi)$. Let $f \in L_{p;w}(-\pi, \pi)$. We have

$$\int_{-\pi}^{\pi} |f(t)|^p w_0(t)\,dt = \int_{-\pi}^{\pi} |f(-t)|^p w_0(-t)\,dt =$$

$$= \int_{-\pi}^{\pi} |f(-t)|^p w_0(t)\,dt < +\infty \Rightarrow f(-t) \in L_{p;w_0}(-\pi, \pi)\,.$$

Consequently, $L_{p;w_0}(-\pi, \pi)$ (at the same time $L_{p;w}(J)$) is a (R)-space. From $w \in A_p(J)$ it follows that $\exists \delta > 0$ such that $L_{p+\delta}(J) \subset L_{p;w}(J)$. From here it directly follows that the set $C_0^\infty(J)$ is dense in $L_{p;w}(J)$. Then according to the Theorem 2 we have the following

Corollary 1. *Let $w \in A_p(J)\,,\, 1 < p < +\infty$ and $w(x) = w(2\pi - x)$ holds, a.e. on J. Then the system $(\mathscr{T})$ forms a basis for $L_{p;w}(J)$, which has the Riesz property.*

4.2. Lebesgue space with variable exponent $L_{p(\cdot)}(J)$

Let $p \in \mathscr{L}(J)\,\&p : J \to \bar{R}_+$. Assume $1 < p^- = ess\inf_J p \le p^+ = ess\sup_J p < +\infty$. Denote

$$I_{p(\cdot)}(f) = \int_J |f(x)|^{p(x)}\,dx.$$

$L_{p(\cdot)}(J)$ is a B-space, which is defined by relation

$$L_{p(\cdot)}(J) = \left\{ f \in \mathscr{L}_0(J) : \|f\|_{L_{p(\cdot)}(J)} < +\infty \right\},$$

where

$$\|f\|_{L_{p(\cdot)}(J)} = \inf\left\{ \lambda > 0 :\, I_{p(\cdot)}\left(\frac{f}{\lambda}\right) \le 1 \right\}.$$

Let the exponent $p(\cdot)$ satisfy the following conditions:

i)$p(x) = p(2\pi - x)$, $\forall x \in J$;

ii)$\forall x; y$: $|x - y| < \frac{1}{2} \Rightarrow |p(x) - p(y)| \le \frac{c}{-\ln|x-y|}$, for some constant $c > 0$ independent of $x; y$.

As before we can easily check that if i) holds, then $L_{p(\cdot)}(J)$ is a (R)-space. Moreover, from ii) it follows that the set $C_0^\infty(J)$ is dense in $L_{p(\cdot)}(J)$ (regarding the related facts one can see, e.g. the monograph [13]). Under the condition ii) from the results of the works [14, 15, 16, 17, 18, 19] it follows that the trigonometric system $\{1; \cos nx; \sin nx\}_{n\in N}$ forms a basis for $L_{p(\cdot)}(J)$. Then the following corollary follows from Theorem 2.

Corollary 2. *Let the exponent $p(\cdot)$ satisfies the conditions i), ii). Then the system $(\mathscr{T})$ forms a basis for $L_{p(\cdot)}(J)$, which has the Riesz Property.*

4.3. The weighted grand Lebesgue space $L_{p);w}(J)$

This a Banach space of $\mathscr{L}_0(J)$ functions with the norm

$$\|f\|_{L_{p);w}(J)} = \sup_{0<\varepsilon<p-1} \left(\varepsilon \int_J |f|^{p-\varepsilon} w dx\right)^{\frac{1}{p-\varepsilon}}, \, 1 < p < +\infty,$$

for appropriate weight function $w : J \to \bar{R}_+$. This is nonseparable space and the closure of $C_0^\infty(J)$ in $L_{p);w}(J)$ denote by $N_{p);w}(J)$. Regarding this separable subspace it is valid the following proper and continuous embeddings

$$L_{p);w}(J) \subset N_{p;w}(J) \subset L_{p-\varepsilon_0;w}(J), \forall \varepsilon_0 \in (0, p-1).$$

In [19] it is proved that, if $w_0 \in A_p$, then the system of exponent $\left\{e^{int}\right\}_{n\in Z}$ (at the same time the trigonometric system (3)) forms a basis for $L_{p);w_0}(-\pi, \pi)$, $1 < p < +\infty$. In addition, if the weight function $w_0(\cdot)$ satisfies $w_0(-x) = w(x)$, a.e. on $(-\pi, \pi)$, then according to the Proposition 1 the system (3) has the Riesz Property in $L_{p);w_0}(-\pi, \pi)$. As a result as a sequence of Theorem 2 we obtain the validity of the following

Corollary 3. *Let the weight function $w \in A_p(J)$, $1 < p < +\infty$, satisfy $w(x) = w(2\pi - x)$, a.e. on J. Then the system $(\mathscr{T})$ forms a basis for $L_{p);w}(J)$ and it has the Riesz Property.*

4.4. Morrey space $L_{p;\lambda}(J)$

This is a Banach space of $\mathscr{L}_0(J)$ functions with the norm

$$\|f\|_{L_{p;\lambda}(J)} = \sup_{I\subset J} \left(\frac{1}{|I|^\lambda} \int_I |f|^p dx\right)^{\frac{1}{p}}, \, 1 < p < +\infty, \, 0 \le \lambda < 1,$$

where sup is taken over all intervals $I \subset J$. For $\lambda : 0 < \lambda < 1$, this space is nonseparable and the closure of $C_0^\infty(J)$ in $L_{p;\lambda}(J)$ denote by $N_{p;\lambda}(J)$ (it is obvious $L_{p;0}(J) = L_p(J)$). It is evident that the space $N_{p;\lambda}(-\pi,\pi)$ is a (R)-space and as established in [20] (see also [21, 22, 23]) the system of exponents $\left\{e^{int}\right\}_{n\in Z}$(at the same time the trigonometric system (3) forms a basis for $N_{p;\lambda}(-\pi,\pi)$, $1 < p < +\infty$, $0 \le \lambda < 1$. As follows from Proposition 1, the system (3) has the Riesz Property in $N_{p;\lambda}(-\pi,\pi)$. Therefore, it is valid

Corollary 4. *The system* $(\mathscr{T})$ *forms a basis for* $N_{p;\lambda}(-\pi,\pi)$, $1 < p < +\infty$, $0 \le \lambda < 1$, *and it has the Riesz Property.*

4.5. Symmetric spaces $X(J)$

Let $X(J)$ be a symmetric space on J with Lebesgue measure m(one dimensional). For $f \in \mathscr{L}_0(J)$ denote $\hat{f}(x) = f(2\pi - x)$, $\forall x \in J$. It is not hard to see that $m_f(\lambda) \equiv m_{\hat{f}}(\lambda)$, $\forall\lambda > 0$, and therefore $\|f\|_{X(J)} = \left\|\hat{f}\right\|_{X(J)}$, i.e. $X(J)$ is a (R)-space. Let α_X and β_X be the Boyd indices (lower and upper, respectively). The closure of $C_0^\infty(J)$ in $X(J)$ denote by $X_s(J)$. It is known that under condition $\alpha_X; \beta_X \in (0,1)$ the trigonometric system (3) forms a basis for $X_s(-\pi,\pi)$ (and it is evident that at the same time in $X_s(J)$). According to the above discussion as follows from Proposition 1, this system has the Riesz Property in $X_s(-\pi,\pi)$ (see e.g. [9, 10]). Thus, from Theorem 2 we obtain

Corollary 5. *Let* $X(J)$ *be a symmetric space with Boyd indices* $\alpha_X; \beta_X \in (0,1)$. *Then the system* $(\mathscr{T})$ *forms a basis in separable subspace* $X_s(J)$ *and it has the Riesz Property.*

Let us recall that Lebesgue space, grand Lebesgue space, Orlicz space, Marcinkiewicz space, Lorentz space and others belong to the class of symmetric spaces. For completeness let us give the Boyd indices of some these spaces. Regarding the calculation of these indices one can see the monographs [9, 10, 11] and the works [25, 26, 27].

1) Lebesgue space $X(J) \equiv L_p(J)$, $1 < p < +\infty$.

$$\alpha_p = \beta_p = \frac{1}{p}.$$

2) Grand Lebesgue space $X(J) \equiv L_{p)}(J)$, $1 < p < +\infty$.

$$\alpha_{p)} = \beta_{p)} = \frac{1}{p}.$$

3) Marcinkiewicz space $M_{p;\lambda}(J)$, $1 < p < +\infty$, $0 \le \lambda < 1$.

This is a Banach space of $\mathscr{L}_0(J)$ functions with the norm

$$\|f\|_{M_{p;\lambda}(J)} = \sup_{I\subset J}\left(\frac{1}{|I|^{\lambda}}\int_I |f|^p\,dx\right)^{\frac{1}{p}},$$

where sup is taken over all measurable (in Lebesgue sense) subsets $I \subset J$. $M_{p;\lambda}(J)$ is also nonseparable, but unlike Morrey space it is a symmetric space. Boyd indices of this space are

$$\alpha_{p;\lambda} = \beta_{p;\lambda} = \frac{1-\lambda}{p}.$$

5. Orlicz space $L_M(J)$.

Let us define this space.

Continuous convex on R function $M(\cdot)$ is called an N-function if it is even and satisfies the conditions

$$\lim_{x\to 0}\frac{M(x)}{x} = 0\,;\ \lim_{x\to\infty}\frac{M(x)}{x} = \infty.$$

Denote by $\mathscr{N}$ the set of all N-functions.

For $M \in \mathscr{N}$ define a complementary to $M(\cdot)$ N-function by relation

$$M^*(x) = \max_{y\geq 0}\left[y\,|x| - M(y)\right],\ \forall x\in R.$$

The function $M \in \mathscr{N}$ satisfies Δ_2-condition for large values of x, if $\exists k > 0\ \&\ \exists x_0 > 0$:

$$M(2x) \leq kM(x)\,,\ \forall x \geq x_0.$$

Let

$$\rho_M(f) = \int_J M(f(x))\,dx.$$

Assume

$$\tilde{L}_M(J) = \{f \in \mathscr{L}_0(J) :\ \rho_M(f) < +\infty\}.$$

$\tilde{L}_M(J)$ is called an Orlicz space. Define Orlicz space $L_M(J)$ by expression

$$L_M(J) = \left\{f \in \mathscr{L}_0(J) :\ |(f;g)| < +\infty\,,\ \forall g \in \tilde{L}_{M^*}(J)\right\},$$

where

$$(f;g) = \int_J f(x)\,\overline{g(x)}dx,$$

$M^*(\cdot)$ is a complementary to$M(\cdot)$ function.

It is known the following

Statement 3. *If $M(\cdot)$ satisfies the Δ_2-condition, then $L_M(J) \equiv \tilde{L}_M(J)$ and the closure of the set of bounded (including continuous) functions coincides with $L_M(J)$.*

Let $M \in \mathscr{L}$ and by $M^{-1}(\cdot)$ denote its inverse function on R_+. Assume

$$h(t) = \limsup_{x\to\infty} \frac{M^{-1}(x)}{M^{-1}(tx)}, \; t > 0.$$

The Boyd indices of $L_M(J)$ are

$$\alpha_M = -\lim_{t\to\infty} \frac{\log h(t)}{\log t}; \beta_M = -\lim_{t\to+0} \frac{\log h(t)}{\log t}.$$

More information concerning above facts one can get from, e.g. [25, 28]

Acknowledgements

This work was supported by the Azerbaijan Science Foundation-Grant No: AEF-MCG-2023-1(43)-13/05/1-M-05.

References

[1] Frankl, F.I. (1956) *Subsonic flow past profile with a supersonic zone ending in a direct shock wave*, Appl. Math. Mech., 20(2), 196-202.

[2] Frankl, F.I. (1973) Selected works on gas dynamics, Moscow, *Nauka*, 711 p.

[3] Bitsadze, A.V., Samarskii, A.A. (1969) *Some elementary generalizations of linear elliptic boundary value problems*, Dokl. Akad. Nauk SSSR, 185(4), 739–740.

[4] Ionkin, N.I., Moiseev, E.I. (1979) *A problem for a heat equation with two-point boundary conditions*, Differ. Uravn., 15(7), 1284–1295.

[5] Moiseev, E.I. (1999) *On the solution of a nonlocal boundary value problem by the spectral method*, Differ. Uravn., 35(8), 1094–1100.

[6] Lerner,M.E., Repin, O.A. (1999) *On Frankl'-type problems for some elliptic equations with degeneration of various types*, Differ. Equ., 35(8), 1098–1104.

[7] Mammadov, T.J. (2023) *Strong Solvability of a Nonlocal Problem for the Laplace Equation in Weighted Grand Sobolev Spaces*, Azerb. J. Math., 13(1), 188-204.

[8] Bilalov, B.T., Nasibova, N.P., Alili, V.Q. (2025) *On the strong solvability of a nonlocal boundary value problem for the Laplace equation in an unbounded domain*, Journal of Contemporary Applied Mathematics, 15(1), 92-106

[9] Bennet, C., Sharpley, R. (1988) Interpolation of operators, *Academic Press.*

[10] Lindenstrauss, J., Tzafriri, L. (1979) Classical Banach Spaces, II, *Springer-Verlag*, 243 p.

[11] Krein, Yu., Petunin, I., E.M. Semenov, (1978) *Interpolation of linear operators*, Moscow, Nauka.

[12] Hunt, R., Muckenhoupt, B., Wheeden, R. (1973) *Weighted Norm Inequalities For The Conjugate Function And Hilbert Transform*, Trans. Amer. Math. Soc., 176, 227-251.

[13] Cruz-Uribe, D.V., Fiorenza, A. (2013) Variable Lebesgue spaces, Birkhauser, *Springer.*

[14] B.T. Bilalov, Z.G. Guseynov, (2008) *Bases from exponent in Lebesgue spaces of functions with variable summability exponents*, Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci., XXVIII(1), 43-48.

[15] Bilalov, B.T., Guseynov, Z.G. (2011) *Basicity criterion for perturbed systems of exponents in Lebesgue spaces with variable summability*, Dokl. Akad. Nauk, 436(5), 586–589.

[16] Bilalov, B.T., Guseynov, Z.G. (2012) *Basicity of a system of exponents with a piece-wise linear phase in variable spaces*, Mediterr. J. Math., 9(3), 487-498.

[17] Sharapudinov, I.I. (2007) *Some problems of approximation theory in the spaces* $L_{p(\cdot)}(E)$, Anal. Math., 33(2), 135-153.

[18] Sharapudinov, I.I. (2014) *On Direct And Inverse Theorems Of Approximation Theory In Variable Lebesgue And Sobolev Spaces*, Azerb. J. Math., 4(1), 55-72.

[19] Zeren, Y., Ismailov, M.I., Sirin, F. (2022) *On the Basis Properties of a System of Eigenfunctions of a Spectral Problem for a Second-Order Discontinuous Differential Operator in the Weighted Grand-Lebesgue Space with a General Weight*, Filomat, 36(17), 6037–6050.

[20] Bilalov, B.T., Guliyeva, A.A. (2014) *On basicity of the perturbed systems of exponents in Morrey-Lebesgue space*, Internat. J. Math., 25:1450054, 1-10.

[21] Bilalov, B.T., Huseynli, A.A., El-Shabrawy, S.R. (2019) *Basis Properties of Trigonometric Systems in Weighted Morrey Spaces*, Azerb. J. Math., 9(2), 200-226.

[22] Bilalov, B.T., Seyidova, F.Sh. (2019) *Basicity of a system of exponents with a piece-wise linear phase in Morrey-type spaces*, Turkish J. Math., 43, 1850–1866.

[23] Bilalov, B.T. (2019) *The basis property of a perturbed system of exponentials in Morrey-type spaces*, Sib. Math. J., 60(2), 323-350.

[24] Formica, M.R., Giova, R. (2015) *Boyd Indices in Generalized Grand Lebesgue Spaces and Applications*, Mediterr. J. Math., 12(3), 987–995.

[25] Boyd, D.W. (1967) *Spaces between a pair of reflexive Lebesgue spaces*, Proc. Amer. Math. Soc., 18(2), 215-219.

[26] Bilalov, B.T., Sadigova, S.R. (2022) *On local solvability of higher order elliptic equations in rearrangement invariant spaces*, Sib. Math. J., 63(3), 425-437

[27] Bilalov, B.T., Sadigova, S.R., Alili, V.G. (2024) *The Method of Boundary Value Problems in the Study of the Basis Properties of Perturbed System of Exponents in Banach Function Spaces*, Comput. Methods Funct. Theory, 24, 101-120.

[28] Reo, M.M., Ren, Z.D. (2002) Applications of Orlicz Spaces, New-York-Basel, 465 p.

Bilal T. Bilalov
Odlar Yurdu University, Baku, Azerbaijan
Institute of Mathematics and Mechanics, The Ministry of Science and Education, Baku, Azerbaijan
Yildiz Technical University, Istanbul, Türkiye
Azerbaijan University of Architecture and Construction, Baku, Azerbaijan
E-mail: b_bilalov@mail.ru

Valid F. Salmanov
Azerbaijan State Oil and Industry University, Baku, Azerbaijan
E-mail: valid.salmanov@mail.ru

Tural Hagverdi
Institute of Mathematics and Mechanics, The Ministry of Science and Education, Baku, Azerbaijan
E-mail: turalhagverdi@gmail.com

Received 14 March 2025
Accepted 28 August 2025