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Determination of source terms in a time-fractional parabolic equation via nonlocal measurements

Elvin I. Azizbayov, Yashar T. Mehraliyev

Abstract. This paper addresses the determination of source terms in a time-fractional parabolic equation using nonlocal measurement data. To this end, the original problem is first transformed into an equivalent formulation with prescribed data and the equivalence between the two problems is rigorously established. Using the Fourier method, the solution of the resulting auxiliary problem is expressed as a system of integral equations. The contraction mapping principle is then applied to prove the existence and uniqueness of the solution to this system. Finally, based on the established equivalence, sufficient conditions ensuring the existence and uniqueness of the classical solution to the original inverse problem are derived.

Key Words and Phrases: Inverse problem, time-fractional parabolic equation, periodic boundary condition, classical solution, Mittag-Leffler type function

2010 Mathematics Subject Classifications: Primary 35R30, 35R11; Secondary 35A01, 35A02, 35A09

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1. Introduction

Fractional differential equations generalize ordinary and partial differential equations to arbitrary fractional orders. Currently, the theory of fractional calculus has become an essential tool in numerous branches of contemporary science. In recent decades, both direct and inverse boundary value problems for time-fractional parabolic equations have garnered significant attention from researchers. The inverse boundary problem for time-fractional parabolic equations has broad applications, particularly in systems where memory effects and anomalous diffusion play a key role. For example, in many physical systems, heat or mass transfer can exhibit anomalous diffusion, which is effectively modelled by fractional differential equations. The inverse boundary problem helps determine unknown parameters, such as the source term or boundary conditions, in systems of fractional order. This approach is especially valuable for modeling materials with memory or systems in which heat or mass diffuses more slowly than predicted by classical diffusion laws.

It is obvious that the inverse boundary value problems for parabolic equations of integer order have been widely studied using various methods and with different boundary conditions (e.g. [3, 4, 7, 10, 11, 13, 14, 15, 17, 18, 19, 23, 24] and references therein). However, inverse boundary value problems for fractional-order parabolic equations have been studied relatively little (e.g. [1, 2, 5, 8, 9, 12, 16, 21, 22, 25], and so on). We briefly review previous studies concerning inverse boundary value problems for time-fractional parabolic equations. The paper [2] considers a nonlocal inverse problem of Bitsadze-Samarskii type for a degenerate fractional-order parabolic equation and proves the uniqueness and existence theorems for the solution to the considered inverse problem. The global theorems on the existence and uniqueness of the solution to the inverse coefficient problem for a fractional diffusion equation with initial nonlocal and integral overdetermination conditions were proved by Durdiev and Rahmonov [9], the authors Ismailov and Çiçek [12] consider the inverse problem of recovering the time-dependent source term in the time-fractional diffusion equation and establish theorems on the existence and uniqueness of the solution. A class of inverse problems involving the restoration of the right-hand side of a fractional heat equation with involution is considered in [25], and results on the existence and uniqueness of solutions to the problems are presented.

It should be noted that the problem statement and the proof techniques used in this paper differ from previously published works. More precisely, the technique used here is based on transforming the original inverse problem into an equivalent one, studying the solvability of the equivalent problem, and then transitioning back to the original problem.

2. Mathematical formulation of the problem

Let $0 < T < +\infty$ be a fixed time moment, and D_T be the rectangular region defined by the inequalities $0 \le x \le 1$ and $0 \le t \le T$. We consider the problem of finding a triple $\{u(x,t), a(t), b(t)\}$ such that the components satisfy the following time-fractional parabolic equation:

$$ {}^C D_{0t}^{\alpha} u(x,t) + = u_{xx}(x,t) + a(t)u(x,t) + b(t)g(x,t) + f(x,t) \quad (x,t) \in D_T, \quad (1)$$

with the initial condition

$$u(x,0) = \varphi(x), \ 0 \le x \le 1, \quad (2)$$

the periodic condition

$$u(0,t) = u(1,t),\ 0 \le t \le T, \tag{3}$$

the nonlocal integral condition

$$\int_0^1 u(x,t)dx = 0,\ 0 \le t \le T, \tag{4}$$

and the overdetermination conditions

$$u(x_i,t) = h_i(t),\ i = 1,2;\ 0 \le t \le T, \tag{5}$$

where x_1 and x_2 are known distinct internal points of the segment $(0,1)$; $f(x,t), g(x,t), h_1(t)$ and $h_2(t)$ are given functions, and ${}^C D_{0t}^{\alpha} u(x,t))$ is Caputo-type fractional derivative of $u(x,t)$ defined by the formula

$${}^C D_{0t}^{\alpha} u(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x,\tau)}{\partial \tau} \frac{d\tau}{(t-\tau)^{\alpha}},\quad \alpha \in (0,1).$$

The notation $\Gamma(\alpha)$ in the last equation represents the Euler gamma function.

Definition 1. *The triple $\{u(x,t), a(t), b(t)\}$ is said to be a classical solution to problem* (1)-(5)*, if the functions $u(x,t), a(t)$ and $b(t)$ satisfy the following conditions:*

i) The function $u(x,t)$, and the derivatives ${}^C D_{0t}^{\alpha} u(x,t)$ and $u_{xx}(x,t)$ are continuous in D_T.

ii) The time dependent functions $a(t)$ and $b(t)$ are continuous on the interval $[0,T]$.

iii) The equation (1) *and conditions* (2)-(5) *are satisfied for the functions $u(x,t)$, $a(t)$, and $b(t)$ in the usual sense.*

The following theorem is valid.

Theorem 1. *Assume that $f(x,t), g(x,t) \in C(D_T)$, $\int_0^1 f(x,t)dx = \int_0^1 g(x,t)dx = 0, 0 \le t \le T$, ${}^C D_{0t}^{\alpha} h_i(t) \in C[0,T]$ $(i = 1,2)$, $h(t) \equiv h_1(t)g(x_2,t) - h_2(t)g(x_1,t) \ne 0$, $0 \le t \le T$ and the compatibility conditions*

$$\int_0^1 \varphi(x)dx = 0, \tag{6}$$

$$h_1(0) = \varphi(x_i) \ (i = 1, 2), \tag{7}$$

holds. Then the problem of finding a classical solution of (1)–(5) *is equivalent to the problem of determining the functions* $u(x,t), a(t)$ *and* $b(t)$*, satisfying the equation* (1)*, the conditions* (2)*,* (3)*, and the relations*

$$u_x(0,t) = u_x(1,t), \ 0 \le t \le T, \tag{8}$$

$${}^C D_{0t}^{\alpha} h_i(t) = u_{xx}(x_i,t) + a(t)h_i(t) + b(t)g(x_i,t) + f(x_i,t), \ i = 1,2, \ 0 \le t \le T. \tag{9}$$

Proof. Let $\{u(x,t), a(t), b(t)\}$ be a classical solution to (1)–(5). Integrating equation (1) from 0 to 1 with respect to x, we find

$${}^C D_{0t}^{\alpha} \int_0^1 u(x,t)dx = u_x(1,t) - u_x(0,t)$$

$$+a(t)\int_0^1 u(x,t)dx + b(t)\int_0^1 g(x,t)dx + \int_0^1 f(x,t)dx, \ 0 \le t \le T. \tag{10}$$

Taking into account the equality

$$\int_0^1 f(x,t)dx = \int_0^1 g(x,t)dx = 0, \ 0 \le t \le T,$$

we arrive at (8). Setting $x = x_i$ in Eq. (1), the procedure yields

$${}^C D_{0t}^{\alpha} u(x_i,t) = u_{xx}(x_i,t) + a(t)u(x_i,t)$$

$$+b(t)g(x_i,t) + f(x_i,t), \ i = 1,2, \ 0 \le t \le T. \tag{11}$$

Since ${}^C D_{0t}^{\alpha} h_i(t) \in C[0,T]$ $(i = 1,2)$, by virtue of overdetermination condition (5), we get

$${}^C D_{0t}^{\alpha} u(x_i,t) = {}^C D_{0t}^{\alpha} h_i(t), \ i = 1,2, \ 0 \le t \le T. \tag{12}$$

Consequently, taking into account (5) and (12), we arrive at (9).

Now assume that the triple $\{u(x,t), a(t), b(t)\}$ is a solution to the problem (1)–(3), (8), (9). Then from (10), taking into account (3) and (8), we obtain

$${}^C D_{0t}^{\alpha} \int_0^1 u(x,t)dx = a(t)\int_0^1 u(x,t)dx, \ 0 \le t \le T. \tag{13}$$

By (2) and (9), it is easy to see that

$$\int_0^1 u(x,0)dx = \int_0^1 \varphi(x)dx = 0. \tag{14}$$

Since problem (13), (14) has only a trivial solution, it follows that

$$\int_0^1 u(x,t)dx = 0 \ (0 \le t \le T),$$

i.e., condition (4), is satisfied.

Next, from (9) and (11), we find

$${}^C D_{0t}^{\alpha}(u(x_i,t) - h_i(t)) = a(t)(u(x_i,t) - h_i(t)), \ i = 1,2; \ 0 \le t \le T. \tag{15}$$

By virtue of (2) and the compatibility condition (9), we have

$$u(x_i,0) - h_i(0) = \varphi(x_i) - h_i(0) = 0, i = 1,2. \tag{16}$$

From (15), taking into account (16), it is clear that condition (5) is also satisfied.

3. Classical solvability of inverse boundary value problem

It is known that [6] the following sequences of functions

$$1, \cos\lambda_1 x, \sin\lambda_1 x, ..., \cos\lambda_k x, \sin\lambda_k x, ... \tag{17}$$

forms a basis in $L_2(0,1)$ for $\lambda_k = 2k\pi$ $(k = 0,1,\ldots)$.

In fact, since the system (12) forms a basis in $L_2(0,1)$, then each solution to problem (1)–(3), (7), (8) can be sought in the form:

$$u(x,t) = \sum_{k=0}^{\infty} u_{1k}(t)\cos\lambda_k x + \sum_{k=1}^{\infty} u_{2k}(t)\sin\lambda_k x, \ \lambda = 2k\pi, \tag{18}$$

where

$$u_{10}(t) = \int_0^1 u(x,t)dx,$$

$$u_{1k}(t)=2\int\limits_0^1 u(x,t)\cos\lambda_k x dx,\ u_{2k}(t)=2\int\limits_0^1 u(x,t)\sin\lambda_k x dx,\ k=1,2,\ldots.$$

Further, using the formal scheme of the Fourier method, from (1) and (2), we get

$$ {}^C D_{0t}^{\alpha} u_{10}(t)=F_{10}(t;u,a,b),\ 0\le t\le T, \tag{19}$$

$$ {}^C D_{0t}^{\alpha} u_{ik}(t)+\lambda_{ik}^2 u_{ik}(t)=F_{ik}(t;u,a,b),\ i=1,2,\ k=1,2,\ldots,\ 0\le t\le T, \tag{20}$$

$$u_{ik}(0)=\varphi_{ik},\ i=1,2,\ k=0,1,2,\ldots, \tag{21}$$

where

$$F_{ik}(t;u,a,b)=f_{ik}(t)+b(t)g_{ik}(t)+a(t)u_{ik}(t),\ i=1,2,\ k=0,1,2,\ldots,$$

$$f_{10}(t)=\int\limits_0^1 f(x,t)dx,\ g_{10}(t)=\int\limits_0^1 g(x,t)dx,$$

$$f_{1k}(t)=2\int\limits_0^1 f(x,t)\cos\lambda_k x dx,\ f_{2k}(t)=2\int\limits_0^1 f(x,t)\sin\lambda_k x dx,\ k=1,2,\ldots,$$

$$g_{1k}(t)=2\int\limits_0^1 g(x,t)\cos\lambda_k x dx,\ g_{2k}(t)=2\int\limits_0^1 g(x,t)\sin\lambda_k x dx,\ k=1,2,\ldots,$$

$$\varphi_{10}=\int\limits_0^1 \varphi(x)dx,\ \varphi_{1k}(t)=2\int\limits_0^1 \varphi(x)\cos\lambda_k x dx,$$

$$\varphi_{2k}(t)=2\int\limits_0^1 \varphi(x)\sin\lambda_k x dx,\ k=1,2,\ldots.$$

Solving the problem (19)–(21), we arrive at

$$u_{10}(t)=\varphi_{10}+\frac{1}{\Gamma(\alpha)}\int\limits_0^t (t-\tau)^{\alpha-1}F_{10}(\tau;u,a,b)d\tau, \tag{22}$$

$$u_{ik}(t)=\varphi_{ik}E_{\alpha}(-\lambda_k^2 t^{\alpha})$$

$$+\int\limits_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k^2(t-\tau)^{\alpha})F_{ik}(\tau;u,a,b)d\tau,\ i=1,2,\ k=1,2,\ldots, \tag{23}$$

whereas $E_{\alpha}(-\lambda_k^2 t^{\alpha})$ and $E_{\alpha,\alpha}(-\lambda_k^2(t-\tau)^{\alpha})$ are Mittag-Leffler functions.

Definition 2. *[22] The generalized Mittag-Leffler function is defined by*

$$E_{\alpha,\beta}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}, \quad z \in C,$$

where $\alpha > 0$ *and* $\beta \in R$.

Remark 1. *[16] Let* $0 < \alpha < 2$, *and* $\beta \in R$ *be arbitrary numbers. If* $\frac{\pi\alpha}{2} < \mu < \min\{\pi, \pi\alpha\}$, *then there exists a constant* $C_{\alpha,\beta}$ *that*

$$|E_{\alpha,\beta}(z)| \le \frac{C_{\alpha,\beta}}{1+z}, \quad \mu \le |\arg(z)| \le \pi.$$

Substituting the expressions $u_k(t)$ $(k = 0, 1, ...)$ described by (23) into (18), to determine the first component of the solution (1)–(3), (8), (9), we have

$$u(x,t) = \varphi_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} F_{10}(\tau; u, a, b) d\tau + \sum_{k=1}^{\infty} \{\varphi_{1k} E_\alpha(-\lambda_k^2 t^\alpha)$$

$$+ \left. \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_k^2 (t-\tau)^\alpha) F_{1k}(\tau; u, a, b) d\tau \right\} \cos\lambda_k x$$

$$+ \sum_{k=1}^{\infty} \{\varphi_{2k} E_\alpha(-\lambda_k^2 t^\alpha)$$

$$+ \left. \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_k^2 (t-\tau)^\alpha) F_{2k}(\tau; u, a, b) d\tau \right\} \sin\lambda_k x. \tag{24}$$

Now, from (9), taking into account (18), we get

$$a(t) = [h(t)]^{-1} \{ ({}^C D_{0t}^\alpha h_1(t) - f(x_1,t)) g(x_2,t) - ({}^C D_{0t}^\alpha h_2(t) - f(x_2,t)) g(x_1,t)$$

$$+ \sum_{k=1}^{\infty} \lambda_k^2 u_{1k}(t) (g(x_2,t) \cos\lambda_k x_1 - g(x_1,t) \cos\lambda_k x_2)$$

$$+ \left. \sum_{k=1}^{\infty} \lambda_k^2 u_{2k}(t) (g(x_2,t) \sin\lambda_k x_1 - g(x_1,t) \sin\lambda_k x_2) \right\}, \tag{25}$$

$$b(t) = [h(t)]^{-1} \{ h_1(t) ({}^C D_{0t}^\alpha h_2(t) - f(x_2,t) - h_2(t)) ({}^C D_{0t}^\alpha h_1(t) - f(x_1,t))$$

$$+\sum_{k=1}^{\infty}\lambda_k^2 u_{1k}(t)(h_1(t)\cos\lambda_k x_2-h_2(t)\cos\lambda_k x_2)$$

$$\left.+\sum_{k=1}^{\infty}\lambda_k^2 u_{2k}(t)(h_1(t)\sin\lambda_k x_2-h_2(t)\sin\lambda_k x_1)\right\},\tag{26}$$

where

$$h(t)\equiv h_1(t)g(x_2,t)-h_2(t)g(x_1,t)\neq 0,\ \ 0\le t\le T.\tag{27}$$

Substituting expression (23) into (25) and (26), we obtain

$$a(t)=[h(t)]^{-1}\{({}^C D_{0t}^{\alpha}h_1(t)-f(x_1,t))g(x_2,t)-({}^C D_{0t}^{\alpha}h_2(t)-f(x_2,t))g(x_1,t)$$

$$+\sum_{k=1}^{\infty}\lambda_k^2\left[\varphi_{1k}E_{\alpha}(-\lambda_k^2 t^{\alpha})+\int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k^2(t-\tau)^{\alpha})F_{1k}(\tau;u,a,b)d\tau\right]$$

$$\times(g(x_2,t)\cos\lambda_k x_1-g(x_1,t)\cos\lambda_k x_2)$$

$$+\sum_{k=1}^{\infty}\lambda_k^2\left[\varphi_{2k}E_{\alpha}(-\lambda_k^2 t^{\alpha})+\int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k^2(t-\tau)^{\alpha})F_{2k}(\tau;u,a,b)d\tau\right]$$

$$\times(g(x_2,t)\sin\lambda_k x_1-g(x_1,t)\sin\lambda_k x_2)\},\tag{28}$$

$$b(t)=[h(t)]^{-1}\{h_1(t)({}^C D_{0t}^{\alpha}h_2(t)-f(x_2,t))-h_2(t)({}^C D_{0t}^{\alpha}h_1(t)-f(x_1,t))$$

$$+\sum_{k=1}^{\infty}\lambda_k^2\left[\varphi_{1k}E_{\alpha}(-\lambda_k^2 t^{\alpha})+\int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k^2(t-\tau)^{\alpha})F_{1k}(\tau;u,a,b)d\tau\right]$$

$$\times(h_1(t)\cos\lambda_k x_2-h_2(t)\cos\lambda_k x_1)$$

$$+\sum_{k=1}^{\infty}\lambda_k^2\left[\varphi_{2k}E_{\alpha}(-\lambda_k^2 t^{\alpha})+\int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k^2(t-\tau)^{\alpha})F_{2k}(\tau;u,a,b)d\tau\right]$$

$$\times(h_1(t)\sin\lambda_k x_2-h_1(t)\sin\lambda_k x_1)\}.\tag{29}$$

The following lemma is proved similarly [20].

Lemma 1. *If* $\{u(x,t),a(t),b(t)\}$ *is any solution of* (1)–(3), (8), (9) *then the functions*

$$u_{10}(t)=\int_0^1 u(x,t)dx,$$

$$u_{1k}(t)=2\int_0^1 u(x,t)\cos\lambda_k x dx,\quad u_{2k}(t)=2\int_0^1 u(x,t)\sin\lambda_k x dx,\ k=1,2,\dots,$$

satisfy the system (22), (23) *on the interval* $[0,T]$.

Corollary 1. *By Lemma 1, we may assert that proving the uniqueness of the solution to problem* (1)–(3), (8), (9) *reduces to proving the uniqueness of the solution to system* (24), (28), (29).

In order to study the problem (1)–(3), (8), (9), we consider the following functional spaces: Let $B^3_{2,T}$ denote the set of all functions of the form

$$u(x,t)=\sum_{k=0}^{\infty} u_{1k}(t)\cos\lambda_k x+\sum_{k=1}^{\infty} u_{2k}(t)\sin\lambda_k x,\ \lambda=2k\pi,$$

considered in domain D_T, whereas the functions $u_{1k}(t)$ $(k=0,1,2,...)$ and $u_{2k}(t)$ $(k=1,2,...)$ contained in the last sum are continuous on $[0,T]$, and

$$\|u_{10}(t)\|_{C[0,T]}+\left(\sum_{k=1}^{\infty}(\lambda_k^3\|u_{1k}(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}+\left(\sum_{k=1}^{\infty}(\lambda_k^3\|u_{2k}(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}<+\infty.$$

In the space $B^3_{2,T}$, the operations of addition and scalar multiplication are defined in the usual way, and the norm is given by the formula

$$\|u(x,t)\|_{B^3_{2,T}}=\|u_{10}(t)\|_{C[0,T]}$$

$$+\left(\sum_{k=1}^{\infty}(\lambda_k^3\|u_{1k}(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}+\left(\sum_{k=1}^{\infty}(\lambda_k^3\|u_{2k}(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}.$$

Furthermore, let E^3_T denote the space consisting of the topological product $B^3_{2,T}\times C[0,T]\times C[0,T]$, which is the norm of the element $z(x,t)=\{u(x,t),a(t),b(t)\}$ defined by the formula

$$\|z(x,t)\|_{B^3_{2,T}}=\|u(x,t)\|_{B^3_{2,T}}+\|a(t)\|_{C[0,T]}+\|b(t)\|_{C[0,T]}.$$

It is known that the spaces $B^3_{2,T}$ and E^3_T are Banach spaces.

In the space E^3_T we consider the operator

$$\Phi(u,a,b)=\{\Phi_1(u,a,b),\Phi_2(u,a,b),\Phi_3(u,a,b)\},$$

where

$$\Phi_1(u,a,b)=\tilde{u}(x,t)\equiv\sum_{k=0}^{\infty}\tilde{u}_{1k}(t)\cos\lambda_k x+\sum_{k=1}^{\infty}\tilde{u}_{2k}(t)\sin\lambda_k x,$$

$$\Phi_2(u,a,b)=\tilde{a}(t),\ \ \Phi_3(u,a,b)=\tilde{b}(t),$$

and the functions $\tilde{u}_{10}(t),\tilde{u}_{ik}(t)$ $(i=1,2;k=0,1,2,...),\tilde{a}(t)$ and $\tilde{b}(t)$ are defined by the right-hand sides of (22), (23), (28), and (29), respectively.

It follows that

$$\left|E_\alpha(-\lambda_k^2 t^\alpha)\right|\leq\frac{M_1}{1+\lambda_k^2 t^\alpha},\quad M_1=const\geq 0,$$

$$\left|E_\alpha(-\lambda_k^2 (t-\tau)^\alpha)\right|\leq\frac{M_2}{1+\lambda_k^2 (t-\tau)^\alpha},\quad 0\leq\tau\leq t,\quad M_2=const\geq 0,$$

$$|g(x_2,t)\cos\lambda_k x_1-g(x_1,t)\cos\lambda_k x_2|\leq\||g(x_2,t)|+|g(x_1,t)|\|_{C[0,T]}\equiv p_1(T),$$

$$|g(x_2,t)\sin\lambda_k x_1-g(x_1,t)\sin\lambda_k x_2|\leq\||g(x_2,t)|+|g(x_1,t)|\|_{C[0,T]}\equiv p_1(T),$$

$$|h_1(t)\cos\lambda_k x_2-h_2(t)\cos\lambda_k x_1|\leq\||h_1(t)|+|h_2(t)|\|_{C[0,T]}\equiv p_2(T),$$

$$|h_1(t)\sin\lambda_k x_2-h_2(t)\sin\lambda_k x_1|\leq\||h_1(t)|+|h_2(t)|\|_{C[0,T]}\equiv p_2(T).$$

Consequently, we obtain

$$\|\tilde{u}_{10}(t)\|_{C[0,T]}\leq|\varphi_{10}|+\frac{T^\alpha}{\alpha\Gamma(\alpha)}[\|a(t)\|_{C[0,T]}\,\|u_{10}(t)\|_{C[0,T]}$$
$$+\|b(t)\|_{C[0,T]}\,\|g_{10}(t)\|_{C[0,T]}+\|f_{10}(t)\|_{C[0,T]}], \tag{30}$$

$$\left(\sum_{k=1}^{\infty}(\lambda_k^3\,\|\tilde{u}_{ik}(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}\leq 2M_1\left(\sum_{k=1}^{\infty}(\lambda_k^3\,|\varphi_{ik}|)^2\right)^{\frac{1}{2}}$$
$$+\frac{2T^\alpha M_2}{\alpha}\left[\left(\sum_{k=1}^{\infty}(\lambda_k^3\,\|f_{ik}(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}+\right.$$
$$+\|a(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^3\,\|u_{ik}(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}$$
$$\left.+\|b(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^3\,\|g_{ik}(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}\right], \tag{31}$$

$$\|\tilde{a}(t)\|_{C[0,T]} \leq \left\|[h(t)]^{-1}\right\|_{C[0,T]}$$
$$\times\{\left\|({}^C D_{0t}^{\alpha} h_1(t) - f(x_1,t))g(x_2,t) - ({}^C D_{0t}^{\alpha} h_2(t) - f(x_2,t))g(x_1,t)\right\|_{C[0,T]}$$
$$+\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}} p_1(T)\sum_{i=1}^{2}\left[M_1\left(\sum_{k=1}^{\infty}(\lambda_k^3\left|\varphi_{ik}\right|)^2\right)^{\frac{1}{2}}\right.$$
$$+\frac{T^{\alpha}M_2}{\alpha}\left[\left(\sum_{k=1}^{\infty}(\lambda_k^3\left\|f_{ik}(t)\right\|_{C[0,T]})^2\right)^{\frac{1}{2}} + \|a(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^3\left\|u_{ik}(t)\right\|_{C[0,T]})^2\right)^{\frac{1}{2}}\right.$$
$$\left.\left.+\|b(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^3\left\|g_{ik}(t)\right\|_{C[0,T]})^2\right)^{\frac{1}{2}}\right]\right]\Bigg\}, \tag{32}$$

$$\left\|\tilde{b}(t)\right\|_{C[0,T]} \leq \left\|[h(t)]^{-1}\right\|_{C[0,T]}$$
$$\times\{\left\|h_1(t)({}^C D_{0t}^{\alpha} h_2(t) - f(x_2,t)) - h_2(t)({}^C D_{0t}^{\alpha} h_1(t) - f(x_1,t))\right\|_{C[0,T]}$$
$$+\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}} p_2(T)\sum_{i=1}^{2}\left[M_1\left(\sum_{k=1}^{\infty}(\lambda_k^3\left|\varphi_{ik}\right|)^2\right)^{\frac{1}{2}}\right.$$
$$+\frac{T^{\alpha}M_2}{\alpha}\left[\left(\sum_{k=1}^{\infty}(\lambda_k^3\left\|f_{ik}(t)\right\|_{C[0,T]})^2\right)^{\frac{1}{2}} + \|a(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^3\left\|u_{ik}(t)\right\|_{C[0,T]})^2\right)^{\frac{1}{2}}\right.$$
$$\left.\left.+\|b(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^3\left\|g_{ik}(t)\right\|_{C[0,T]})^2\right)^{\frac{1}{2}}\right]\right]\Bigg\}. \tag{33}$$

Suppose that the following conditions hold for the data of problem (1)–(3), (8), (9):

H_1) $\varphi(x) \in C^2[0,1],\ \varphi'''(x) \in L_2(0,1)$,
$\varphi(0) = \varphi(1), \varphi'(0) = \varphi'(1), \varphi''(0) = \varphi''(1)$;

H_2) $f(x,t), f_x(x,t), f_{xx}(x,t) \in C^2(D_T),\ f_{xxx}(x,t) \in C([0,T]; L_2(0,1))$,
$f(0,t) = f(1,t), f_x(0,t) = f_x(1,t), f_{xx}(0,t) = f_{xx}(1,t),\ 0 \leq t \leq T$;

H_3) $g(x,t), g_x(x,t), g_{xx}(x,t) \in C^2(D_T),\ g_{xxx}(x,t) \in C([0,T]; L_2(0,1))$,
$g(0,t) = g(1,t), g_x(0,t) = g_x(1,t), g_{xx}(0,t) = g_{xx}(1,t),\ 0 \leq t \leq T$;

H_4) $h_i(t),\ {}^C D_{0t}^{\alpha} h_i(t) \in C[0,T],\ i = 1,2$,
$h(t) \equiv h_1(t)g(x_2,t) - h_2(t)g(x_1,t) \neq 0,\ 0 \leq t \leq T$.

Then, from (30)–(32), respectively, we obtain

$$\|\tilde{u}(x,t)\|_{B^3_{2,T}} \leq A_1(T) + B_1(T)\,\|a(t)\|_{C[0,T]}\,\|u(x,t)\|_{B^3_{2,T}} + D_1(T)\,\|b(t)\|_{C[0,T]}\,, \tag{34}$$

$$\|\tilde{a}(t)\|_{C[0,T]} \leq A_2(T)+B_2(T)\,\|a(t)\|_{C[0,T]}\,\|u(x,t)\|_{B^3_{2,T}}+D_2(T)\,\|b(t)\|_{C[0,T]}\,, \tag{35}$$

$$\left\|\tilde{b}(t)\right\|_{C[0,T]} \leq A_3(T)+B_3(T)\,\|a(t)\|_{C[0,T]}\,\|u(x,t)\|_{B^3_{2,T}}+D_3(T)\,\|b(t)\|_{C[0,T]}\,, \tag{36}$$

where

$$A_1(T) = \|\varphi(x)\|_{L_2(0,1)} + \frac{T^\alpha}{\alpha\Gamma(\alpha)}\left\|\|f(x,t)\|_{L_2(0,1)}\right\|_{C[0,T]}$$

$$+4M_1\left\|\varphi'''(x)\right\|_{L_2(0,1)} + \frac{4T^\alpha M_2}{\alpha}\left\|\|f_{xxx}(x,t)\|_{L_2(0,1)}\right\|_{C[0,T]},$$

$$B_1(T) = \frac{T^\alpha}{\alpha\Gamma(\alpha)} + \frac{2T^\alpha M_2}{\alpha},$$

$$D_1(T) = \frac{T^\alpha}{\alpha\Gamma(\alpha)}\left\|\|g(x,t)\|_{L_2(0,1)}\right\|_{C[0,T]} + \frac{4T^\alpha M_2}{\alpha}\left\|\|g_{xxx}(x,t)\|_{L_2(0,1)}\right\|_{C[0,T]},$$

$$A_2(T) = \left\|[h(t)]^{-1}\right\|_{C[0,T]}$$

$$\times\{\left\|({}^C D^\alpha_{0t}h_1(t) - f(x_1,t))g(x_2,t) - ({}^C D^\alpha_{0t}h_2(t) - f(x_2,t))g(x_1,t)\right\|_{C[0,T]}$$

$$+\left(\sum_{k=1}^\infty \lambda_k^{-2}\right)^{\frac{1}{2}} p_1(T)$$

$$\times\left(M_1\left\|\varphi'''(x)\right\|_{L_2(0,1)} + \frac{T^\alpha M_2}{\alpha}\left\|\|f_{xxx}(x,t)\|_{L_2(0,1)}\right\|_{C[0,T]}\right)\Bigg\},$$

$$B_2(T) = \left\|[h(t)]^{-1}\right\|_{C[0,T]} p_1(T)\left(\sum_{k=1}^\infty \lambda_k^{-2}\right)^{\frac{1}{2}}\frac{T^\alpha M_2}{\alpha},$$

$$D_2(T) = \left\|[h(t)]^{-1}\right\|_{C[0,T]} p_1(T)\left(\sum_{k=1}^\infty \lambda_k^{-2}\right)^{\frac{1}{2}}\frac{2T^\alpha M_2}{\alpha}\left\|\|g_{xxx}(x,t)\|_{L_2(0,1)}\right\|_{C[0,T]},$$

$$A_3(T) = \left\|[h(t)]^{-1}\right\|_{C[0,T]}$$

$$\times\{\left\|h_1(t)({}^C D^\alpha_{0t}h_2(t) - f(x_2,t)) - h_2(t)({}^C D^\alpha_{0t}h_1(t) - f(x_+,t))\right\|_{C[0,T]}$$

$$+2\left(\sum_{k=1}^\infty \lambda_k^{-2}\right)^{\frac{1}{2}} p_2(T)$$

$$\times\left(M_1\left\|\varphi'''(x)\right\|_{L_2(0,1)}+\frac{T^\alpha M_2}{\alpha}\left\|\|f_{xxx}(x,t)\|_{L_2(0,1)}\right\|_{C[0,T]}\right)\Bigg\},$$

$$B_3(T)=\left\|[h(t)]^{-1}\right\|_{C[0,T]}p_2(T)\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\frac{T^\alpha M_2}{\alpha},$$

$$D_3(T)=\left\|[h(t)]^{-1}\right\|_{C[0,T]}p_2(T)\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\frac{2T^\alpha M_2}{\alpha}\left\|\|g_{xxx}(x,t)\|_{L_2(0,1)}\right\|_{C[0,T]}.$$

From inequalities (34)–(36), we conclude that

$$\|\tilde{u}(x,t)\|_{B^3_{2,T}}+\|\tilde{a}(t)\|_{C[0,T]}+\left\|\tilde{b}(t)\right\|_{C[0,T]}$$
$$\leq A(T)+B(T)\|a(t)\|_{C[0,T]}\|u(x,t)\|_{B^3_{2,T}}+D(T)\|b(t)\|_{C[0,T]}, \tag{37}$$

where

$$A(T)=A_1(T)+A_2(T)+A_3(T),\quad B(T)=B_1(T)+B_2(T)+B_3(T),$$
$$D(T)=D_1(T)+D_2(T)+D_3(T).$$

Let us prove the following theorem.

Theorem 2. *Let the conditions $H_1)-H_4)$ and the condition*

$$(B(T)(A(T)+2)+D(T))(A(T)+2)<1, \tag{38}$$

be fulfilled. Then, problem (1)–(3), (8), (9) *has a unique solution in the ball $K=K_R$ $(\|z\|_{E^3_T}\leq R\leq A(T)+2)$.*

Proof. First, we write the system of equations (24), (28), (29) in the operator form

$$z=\Phi z, \tag{39}$$

where $z=\{u,a,b\}$. The components $\Phi_i(u,a,b)$ $(i=1,2,3)$, of operator $\Phi(u,a,b)$ defined by the right side of equations (24), (28) and (29), respectively.

Further, consider the operator $\Phi(u,a,b)$ in the ball $K=K_R$ of the space E^3_T and show that the operator Φ takes the elements of the ball $K=K_R$ into itself. Similar to (35), we obtain that for any $z_1,z_2,z_3\in K_R$ the following estimates holds

$$\|\Phi z\|_{E^3_T}\leq A(T)+B(T)\|a(t)\|_{C[0,T]}\|u(x,t)\|_{B^3_{2,T}}+D(T)\|b(t)\|_{C[0,T]}$$

$$\leq A(T) + B(T)(A(T)+2)^2 + D(T)(A(T)+2) < A(T)+2, \tag{40}$$

$$\|\Phi z_1 - \Phi z_2\|_{E_T^3} \leq B(T)R(\|a_1(t)-a_2(t)\|_{C[0,T]} + \|u_1(x,t)-u_2(x,t)\|_{B_{2,T}^3})$$

$$+D(T)\,\|b_1(t)-b_2(t)\|_{C[0,T]}\,. \tag{41}$$

Then by (38), from last two estimates follows that the operator Φ acts in a ball $K = K_R$ and it can be show that the operator Φ is contractive. Therefore the operator Φ has a unique fixed point $\{z\} = \{u, a, b\}$ in the ball $K = K_R$, which is a unique solution of equation (39); i.e. $\{z\} = \{u, a, b\}$ is a unique solution of the systems (24), (28), (29) in the ball $K = K_R$.

Thus, we obtain that the function $u(x,t)$, as an element of the space $B_{2,T}^3$ is continuous and has continuous derivatives $u_x(x,t)$ and $u_{xx}(x,t)$ in D_T.

By virtue of (20), for each $k \in$ N the function ${}^C D_{0t}^{\alpha} u_k(t)$ is continuous on $[0,T]$. Consequently, the following relation holds

$$\left(\sum_{k=1}^{\infty} (\lambda_k {}^C D_{0t}^{\alpha} u_{ik}(t))^2\right)^{\frac{1}{2}} \leq \sqrt{2}\left[\left(\sum_{k=1}^{\infty} (\lambda_k^3 \|u_{ik}(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}\right.$$

$$\left. +2\left\|\|f_x(x,t) + a(t)u_x(x,t) + b(t)g_x(x,t)\|_{C[0,T]}\right\|_{L_2(0,1)}\right] < +\infty.$$

Then, on the basis of the last relation we can write ${}^C D_{0t}^{\alpha} u(x,t) \in C(D_T)$.

It is not hard to verify that equation (1) and conditions (2), (3), (8), and (9) are satisfied in the usual sense. Thus, the triple of $\{u(x,t), a(t), b(t)\}$ is solution to the problem (1)–(3), (8), (9) and due to the assertion of the Lemma 1 it is unique in the ball $K = K_R$.

Hence, from Theorem 2, by virtue of Theorem 1, it follows the validity of the following

Theorem 3. *Let all assertions of Theorem 2 and compatibility conditions* (6), (7) *be satisfied. In addition, let the equality*

$$\int_0^1 f(x,t)dx = \int_0^1 g(x,t)dx = 0, \ \ 0 \leq t \leq T$$

holds. Then problem (1)–(5) *has a unique classical solution in the ball* $K = K_R$ $(\|z\|_{E_T^3} \leq R = A(T)+2)$ *of the space* E_T^3.

4. Conclusions

In this paper, the problem of determining source terms in a time-fractional parabolic equation from nonlocal measurement data has been investigated. The original inverse boundary value problem, formulated under periodic and integral conditions in a finite domain, was first transformed into an equivalent problem with prescribed data, and the equivalence of these formulations was rigorously justified. By applying the Fourier method, the auxiliary problem was reduced to a system of integral equations. The contraction mapping principle was then employed to establish the existence and uniqueness of the solution to this system. Using this equivalence, sufficient conditions ensuring the existence and uniqueness of the classical solution to the original inverse problem were derived.

The results obtained contribute to the theoretical foundation of inverse problems for fractional parabolic equations and may serve as a basis for further research on the development of analytical and numerical methods for the reconstruction of unknown sources in fractional diffusion processes.

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Elvin I. Azizbayov
Academy of Public Administration under the President of the Republic of Azerbaijan, Az1001, Baku, Azerbaijan
E-mail: eazizbayov@dia.edu.az

Yashar T. Mehraliyev
Baku State University, Az1148, Baku, Azerbaijan
E-mail: yashar_aze@mail.ru

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