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q-modified Sumudu transform and its applications

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Abstract. In this work, we explore the concept of q- modified Sumudu transform with its properties. We also give some theorems on q- modified Sumudu transform. Furthermore, we obtain some applications of q- modified Sumudu transform for solving differential equations ("ODEs") with initial and boundary conditions.

Key Words and Phrases: q-calculus, q-modified Sumudu transform, q-integration, q-differentiation.

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1. Introduction

Analysts are effectively locked in fundamentally change to grow the concern as its significance in depicting and examining physical framework [1] to [12] and for understanding the dynamic and delayed differential equations. [13, 14] Jackson [15] presented q-calculus, and presently a day's q-calculus got much noteworthiness in numerous branches of science and designing. The concept of q-calculus is appropriate in fractional calculus, control issues, and finding arrangements of q-difference and q-integral conditions [16] to [19]

The modified Sumudu transform of a function $\hbar(\tau)$ in Ugar [7] is defined by

$$G(\kappa; a) = S_a[\hbar(\tau)] = \frac{1}{\kappa}\int_0^\infty a^{-\frac{\tau}{\kappa}}\hbar(\tau)d\tau. \tag{1}$$

2. Preliminaries

We recall some well known definition and notation used in [16] to [21].

The q-shifted factorials for $q \in (0,1)$ and $\kappa \in \mathbb{C}$ are defined as

$$(\kappa;q)_0 = 1,\ (\kappa;q)_n = \prod_{k=0}^{n-1}(1-\kappa q^k),\ n = 1, 2...,$$

$$(\kappa;q)_\infty = \lim_{n\to\infty}(\kappa;q)_n = \prod_{k=0}^{\infty}(1-\kappa q^k).$$

Also we write

$$[\kappa]_q = \frac{1-q^{\kappa}}{1-q}, \; [\kappa]_q! = \frac{(q;q)_n}{(1-q)^n}, \; n \in \mathbb{N}.$$

The q-derivatives $D_q\hbar$ and D_q^+ of a function $\hbar$, given by [16]

$$(D_q\hbar)(\alpha) = \frac{\hbar(\alpha) - \hbar(q\alpha)}{(1-q)\alpha}, \text{ for } (\alpha \neq 0),$$

$$(D_q\hbar)(0) = \hbar'(0) \text{ exists.}$$

The q-derivative of the product

$$D_q(\hbar.\varphi)(\alpha) = \varphi(\alpha)D_q\hbar(\alpha) + \hbar(q\alpha)D_q\varphi(\alpha).$$

The q-Jackson integral in [15]

$$\int_0^{\kappa} \hbar(\alpha)d_q\alpha = (1-q)\alpha \sum_{n=0}^{\infty} \hbar(\alpha q^n)q^n,$$

$$\int_0^{\infty} \hbar(\alpha)d_q\alpha = (1-q) \sum_{n=-\infty}^{\infty} \hbar(q^n)q^n,$$

provided these sums converge absolutely.

A q-analogue of integration is

$$\int_{\kappa}^{\varpi} \varphi(\alpha)D_q\hbar(\alpha)d_q\alpha = \hbar(\varpi)\varphi(\varpi) - \hbar(\kappa)\varphi(\kappa) - \int_{\kappa}^{\varpi} \hbar(q\alpha)D_q\varphi(\alpha)\; d_q\alpha.$$

In [19] and [24], we have

$$E_q^{\rho} = \sum_{n=0}^{\infty} q^{\frac{n(n-1)}{2}} \frac{\rho^n}{[n]_q!} = (-(1-q)z;q)_{\infty}, \tag{2}$$

$$e_q^{\rho} = \sum_{n=0}^{\infty} \frac{\rho^n}{[n]_q!} = \frac{1}{((1-q)\rho;q)_{\infty}}, \quad |z| < \frac{1}{1-q}. \tag{3}$$

The above equations (2) and (3) satisfy the following equations

$$D_q e_q^{\rho} = e_q^{\rho}, \quad D_q E_q^{\rho} = E_q^{q\rho},$$

and

$$e_q^{\rho} E_q^{-\rho} = E_q^{-\rho} e_q^{\rho} = 1.$$

In [15, 20, 25, 26, 27], we have

$$\Gamma(\vartheta)=\int_0^{\infty}\alpha^{\vartheta-1}e^{-\alpha}d\alpha \quad \text{by}$$

$$\Gamma_q(\vartheta)=\frac{(q;q)_\infty}{(q^\vartheta;q)_\infty}(1-q)^{\vartheta-1}, \qquad \vartheta\neq 0,-1,-2.......$$

Therefore

$$\Gamma_q(\vartheta+1)=[\vartheta]_q\Gamma(\vartheta),\ \Gamma_q(1)=1,$$

and

$$\lim_{q\to 1^-}\Gamma_q(\vartheta)=\Gamma(\vartheta), \quad Re(\vartheta)>0.$$

The function Γ_q has the following q -integral representations

$$\begin{aligned}\Gamma_q(\gamma)&=\int_0^{\frac{1}{1-q}}\vartheta^{\gamma-1}E_q^{-q\vartheta}d_q\vartheta\\ &=\int_0^{\frac{\infty}{1-q}}\vartheta^{\gamma-1}E_q^{-q\vartheta}d_q\vartheta.\end{aligned}$$

The q-integral representation Γ_q is defined in [19, 20, 27, 28, 29] as follows:
For all $\gamma,\vartheta>0$, we have

$$\Gamma_q(\gamma)=K_q(\xi)\int_0^{\frac{\infty}{1-q}}\alpha^{\gamma-1}e_q^{-\alpha}d_q\alpha,$$

and

$$B_q(\vartheta,\gamma)=K_q(\vartheta)\int_0^{\infty}\alpha^{\vartheta-1}\frac{(-\alpha q^{\gamma+1};q)_\infty}{(-\alpha;q)_\infty}d_q\alpha,$$

where,

$$K_q(\vartheta) \quad = \frac{(-q,-1;q)_\infty}{(-q^\vartheta,-q^{1-\vartheta};q)_\infty}.$$

If $\dfrac{\log(1-q)}{\log(q)}\in\mathbb{Z}$, we obtain

$$\Gamma_q(\gamma)=K_q(\gamma)\int_0^{\frac{\infty}{1-q}}\alpha^{\gamma-1}e_q^{-\alpha}d_q\alpha=\int_0^{\frac{\infty}{1-q}}\vartheta^{\gamma-1}E_q^{-q\vartheta}d_q\vartheta.$$

3. Main results

Definition 3.1. The q-modified Sumudu transform of a function $h(\tau)$ is defined by

$$G(\kappa; a) =_q S_a[\hbar(\tau)] = \frac{1}{\kappa\ (1-q)} \int_0^\infty a_q^{-\frac{\tau}{\kappa}}\ \hbar(\tau)\ d_q\tau. \tag{4}$$

Property 3.1. (Linearity property)

If $\hbar(\tau)$ and $\wp(\tau)$ are two functions whose q-modified Sumudu transform exists, then for any constant M and N, we have

$$_qS_a[M\hbar(\tau) + N\wp(\tau)] = M_qS_a[\hbar(\tau)] + N_qS_a[\wp(\tau)].$$

Proof. We can easily write

$$\begin{aligned} _qS_a[M\hbar(\tau) + N\wp(\tau)] &= \frac{1}{(1-q)\ \kappa} \int_0^\infty a_q^{-\frac{\tau}{\kappa}} [M\hbar(\tau) + N\wp(\tau)]\ d_q\tau \\ &= \frac{M}{(1-q)\kappa} \int_0^\infty a_q^{-\frac{\tau}{\kappa}}\ \hbar(\tau)\ d_q\tau + \frac{N}{(1-q)\kappa} \int_0^\infty a_q^{-\frac{\tau}{\kappa}}\ \wp(\tau)\ d_q\tau \\ &= M_qS_a[\hbar(\tau)] + N_qS_a[\wp(\tau)]. \end{aligned}$$

Property 3.2. q- modified Sumudu transform holds for the following:

(1) Let $\hbar(\tau) = 1$, then

$$\begin{aligned} _qS_a(1) &= \frac{1}{\kappa\ (1-q)} \int_0^\infty a_q^{-\frac{\tau}{\kappa}} d_q\tau \\ &= \frac{1}{(1-q)\ \kappa} \int_0^\infty e_q^{-\frac{\tau\ loga}{\kappa}}\ d_q\tau \\ &= \frac{1}{(1-q)\ loga}. \end{aligned}$$

(2) Let $\hbar(\tau) = e_q^{\Omega\tau}$, then

$$\begin{aligned} _qS_a[e_q^{\Omega\tau}] &= \frac{1}{\kappa\ (1-q)} \int_0^\infty e_q^{\Omega\tau}\ e_q^{-\frac{\tau\ loga}{\kappa}}\ d_q\tau \\ &= \frac{1}{\kappa\ (1-q)} \int_0^\infty e_q^{\tau\left(\Omega - \frac{loga}{\kappa}\right)}\ d_q\tau \\ &= \frac{1}{(1-q)\ (loga - \Omega\kappa)}. \end{aligned}$$

(3) Let $\hbar(\tau) = \tau$, then

$$\begin{aligned} _qS_a(\tau) &= \frac{1}{\kappa(1-q)} \int_0^\infty \tau\ e_q^{-\frac{\tau\ loga}{\kappa}}\ d_q\tau \\ &= \frac{\kappa}{(1-q)(loga)^2}. \end{aligned}$$

(4) $$ {}_qS_a(sin_q(\Omega\tau)) = \frac{1}{\kappa\ (1-q)} \int_0^\infty e_q^{-\frac{\tau\ loga}{\kappa}} \Big(\frac{e_q^{i\Omega\tau} - e_q^{-i\Omega\tau}}{2i}\Big)\ d_q\tau $$
$$ = \frac{1}{2i\ u\ (1-q)} \int_0^\infty e^{\tau\Big[i\Omega - \frac{loga}{\kappa}\Big]}\ d_qt - \frac{1}{2i\ \kappa\ (1-q)} \int_0^\infty e^{-\tau\Big[i\Omega + \frac{loga}{\kappa}\Big]}\ d_q\tau $$
$$ = \frac{1}{2i\ \kappa\ (1-q)} \Big[\frac{1}{loga - i\Omega\kappa} - \frac{1}{loga + i\Omega\kappa}\Big] $$
$$ = \frac{\Omega\kappa}{(1-q)\ [(loga)^2 + \Omega^2\kappa^2]}. $$

(5) $$ {}_qS_a(cos_q(\Omega\tau)) = \frac{1}{\kappa\ (1-q)} \int_0^\infty e_q^{-\frac{\tau\ loga}{\kappa}} \Big(\frac{e_q^{i\Omega\tau} + e_q^{-i\Omega\tau}}{2i}\Big)\ d_q\tau $$
$$ = \frac{1}{2i\ \kappa\ (1-q)} \int_0^\infty e^{\tau\Big[i\Omega - \frac{loga}{\kappa}\Big]}\ d_q\tau + \frac{1}{2i\ \kappa\ (1-q)} \int_0^\infty e^{-\tau\Big[i\Omega + \frac{loga}{\kappa}\Big]}\ d_q\tau $$
$$ = \frac{1}{2i\ \kappa\ (1-q)} \Big[\frac{1}{loga - i\Omega\kappa} + \frac{1}{loga + i\Omega\kappa}\Big] $$
$$ = \frac{loga}{(1-q)\ [(loga)^2 + \Omega^2\kappa^2]}. $$

(6) $$ {}_qS_a(sinh_q(\Omega\tau)) = \frac{1}{\kappa\ (1-q)} \int_0^\infty e_q^{-\frac{\tau\ loga}{\kappa}} \Big(\frac{e_q^{\Omega\tau} - e_q^{-\Omega\tau}}{2}\Big)\ d_q\tau $$
$$ = \frac{1}{2\kappa\ (1-q)} \int_0^\infty e^{\tau\Big[\Omega - \frac{loga}{\kappa}\Big]}\ d_q\tau - \frac{1}{2\ \kappa\ (1-q)} \int_0^\infty e^{-\tau\Big[\Omega + \frac{loga}{\kappa}\Big]}\ d_q\tau $$
$$ = \frac{1}{2\ \kappa\ (1-q)(\Omega\kappa - loga)} + \frac{1}{2\ \kappa\ (1-q)(\Omega\kappa + loga)} $$
$$ = \frac{\Omega\kappa}{(1-q)\ [\Omega^2\kappa^2 - (loga)^2]}. $$

(7) $$ {}_qS_a(cosh_q(\Omega\tau)) = \frac{1}{\kappa\ (1-q)} \int_0^\infty e_q^{-\frac{\tau\ loga}{\kappa}} \Big(\frac{e_q^{\Omega\tau} + e_q^{-\Omega\tau}}{2}\Big)\ d_q\tau $$
$$ = \frac{1}{2u\ (1-q)} \int_0^\infty e^{\tau\Big[\Omega - \frac{loga}{\kappa}\Big]}\ d_q\tau + \frac{1}{2\ \kappa\ (1-q)} \int_0^\infty e^{-\tau\Big[\Omega + \frac{loga}{\kappa}\Big]}\ d_q\tau $$
$$ = \frac{1}{2\ \kappa\ (1-q)(\Omega\kappa - loga)} - \frac{1}{2\ \kappa\ (1-q)(\Omega\kappa + loga)} $$
$$ = \frac{loga}{(1-q)\ [\Omega^2\kappa^2 - (loga)^2]}. $$

Property 3.3. (Change of scale property)

If ${}_qS_a\{F(\tau)\} = \hbar(\kappa)$, then ${}_qS_a\{F(\varsigma\ \tau)\} = \frac{1}{\varsigma}\hbar\Big(\frac{loga}{\varsigma\ \tau}\Big)$.

Proof. By definition, we have

$$ {}_qS_a[F(\varsigma\tau)] = \frac{1}{\kappa(1-q)} \int_0^\infty e_q^{-\frac{\tau\ loga}{\kappa}}\ F(\varsigma\tau) d_q\tau $$

putting $\varsigma\tau = \chi \Rightarrow \varsigma d_q\tau = d_q\chi$

$$= \frac{1}{\varsigma\ (1-q)} \int_0^\infty e_q^{-\dfrac{\chi\ loga}{\varsigma\ \kappa}}\ F(\chi)\ d_q\chi$$
$$= \frac{1}{\varsigma}\hbar\Big(\frac{loga}{\varsigma\ \kappa}\Big).$$

Property 3.4. (First shifting property)

Let ${}_qS_a[\hbar(\tau)] = G(\kappa; a)$. Each of the following

$${}_qS_a[a^{\Omega\tau}\hbar(\tau)] = G\Big(\frac{\kappa}{1-\Omega\kappa}\Big).$$

Proof. ${}_qS_a[a^{\Omega\tau}\hbar(\tau)] = \dfrac{1}{\kappa(1-q)}\displaystyle\int_0^\infty a^{\Omega\tau}\ a^{-\dfrac{\tau}{\kappa}}\ d_q\tau$

$$= \frac{1}{\kappa(1-q)}\int_0^\infty a^{\tau\left(\Omega-\dfrac{1}{\kappa}\right)} d_q\tau$$
$$= G\Big(\frac{\kappa}{1-\Omega\kappa}\Big).$$

Property 3.5. (Second shifting property)
If $\hbar(\tau)$ is such that ${}_qS_a\{\hbar(\tau)\} = F(\gamma; a)$, then

$${}_qS_a[\hbar(\tau-\omega)\kappa_\omega(\tau)] = a_q^{-\dfrac{\omega}{\kappa}}\ {}_qS_a[\hbar(\tau)].$$

Proof. We have

$${}_qS_a[\hbar(\tau-\omega)\kappa_\omega(\tau)] = \frac{1}{\kappa\ (1-q)}\int_0^\infty a_q^{-\dfrac{\tau}{\kappa}}\ \hbar(\tau-\omega)\ d_q\tau.$$

Putting $\tau - \omega = M \rightarrow d_q\tau = d_qM$, therefore

$${}_qS_a[f(\tau-\omega)u_\omega(\tau)] = \frac{1}{\kappa\ (1-q)}\int_0^\infty a_q^{-\dfrac{\tau}{\kappa}}\ \hbar(\tau-\omega)\ d_q\tau = \frac{1}{\kappa\ (1-q)}\int_\omega^\infty a_q^{-\dfrac{\omega+M}{\kappa}}\ F(M)\ d_qM$$
$$= a_q^{-\dfrac{\omega}{\kappa}}\ {}_qS_a[\hbar(\tau)].$$

Property 3.6. (Convolution Theorem - q-modified Sumudu transform convolution product:

Definition 3.2: The convolution of $\hbar(\tau)$ and $\wp(\tau)$ is defined by

$$(\hbar * \wp)(\tau) = \frac{1}{(1-q)}\int_0^\tau \hbar(\varpi)\ \wp(\tau-\varpi)\ d_q\varpi.$$

Theorem 3.1 Let ${}_qS_a\{\hbar(\tau)\} = F(\iota; a)$ and ${}_qS_a\{\wp(\tau)\} = G(\iota; a)$ be such that $\hbar(\tau)$ *and* $\wp(\tau)$ be two functions on $(0, \infty)$. Then

${}_qS_a(\hbar * \wp)(\tau) = \kappa\ F(\iota; a)\ G(\iota : a).$

Proof.

$$
\begin{aligned}
{}_qS_a[(\hbar * \wp)(\tau)] &= \frac{1}{\kappa\ (1-q)} \int_0^\infty a_q^{-\frac{\tau}{\kappa}}\ (\hbar * \wp)(\tau)\ d_q\tau \\
&= \frac{1}{\kappa\ (1-q)^2} \int_0^\infty a_q^{-\frac{\tau}{\kappa}} \int_0^\tau (\hbar(\varpi)\ \wp(\tau-\varpi)\ d_q\varpi)\ d_q\tau.
\end{aligned}
$$

Setting $\tau - \varpi = \gamma \Rightarrow d_q\tau = d_q\gamma$, we have

$$
\begin{aligned}
&= \frac{1}{\kappa\ (1-q)^2} \int_0^\infty \int_0^\infty a^{-\frac{\gamma+\varpi}{\kappa}}\ \hbar(\varpi)\ \wp(\gamma)\ d_q\gamma\ d_q\varpi \\
&= \kappa \Big(\frac{1}{\kappa(1-q)} a_q^{-\frac{\varpi}{\kappa}}\ \hbar(\varpi)\ d_q\varpi \Big) \Big(\frac{1}{\kappa(1-q)} a_q^{-\frac{\gamma}{\kappa}}\ \wp(\gamma)\ d_q\gamma \Big) \\
&= \kappa\ F(\iota; a)\ G(\iota : a).
\end{aligned}
$$

Theorem 3.2. If the q-modified Sumudu transform of a function $\hbar(\tau)$ exists, then

${}_qS_a\{\hbar(\tau - N)\ \varphi(\tau - N)\} = e_q^{-\chi\ N} S_a\{\hbar(\tau - N)\}\ \tau > N,$

where $H(\tau)$ is Heaviside unit step function defined by $H(\tau - N) = 1$, when

$\tau > N$ and $H(\tau - N) = 0$, when $\tau < N$.

Proof. We have

$$
\begin{aligned}
{}_qS_a\{\hbar(\tau - N)\ \varphi(\tau - N)\} &= \frac{1}{\kappa\ (1-q)} \int_0^\infty e_q^{-\frac{\tau\ loga}{\kappa}}\ \hbar(\tau - N)\ H(\tau - N)\ d_q\tau \\
&= \frac{1}{\kappa\ (1-q)} \int_0^\infty e_q^{-\frac{\tau\ loga}{\kappa}}\ \hbar(\tau - N)\ d_q\tau,\ \tau > N.
\end{aligned}
$$

By Putting $B = \tau - N \Rightarrow \tau = N + B$

$$
\begin{aligned}
{}_qS_a\{\hbar(B)\ \varphi(B)\} &= \frac{1}{\kappa\ (1-q)} \int_0^\infty e_q^{-\frac{(N+B)\ loga}{\kappa}}\ \hbar(B)\ d_qB \\
&= \frac{1}{\kappa\ (1-q)}\ e_q^{-\frac{N\ loga}{\kappa}} \int_0^\infty e_q^{-\frac{B\ loga}{\kappa}}\ \hbar(B)\ d_qB \\
&= e_q^{-\frac{N\ loga}{\kappa}}\ {}_qS_a\{\hbar(B)\}.
\end{aligned}
$$

Theorem 3.3. If q- modified Sumudu transform of the $\hbar(\kappa)$ exists where $\hbar(\kappa)$ is a periods function of period M (that is $\hbar(\kappa + M) = (\hbar(\kappa)), \forall\kappa$)

$$ {}_qS_a[\hbar(\tau)] = \frac{\left[1 - e_q^{-\frac{M\ loga}{\kappa}}\right]^{-1}}{\kappa(1-q)} \int_0^M e_q^{-\frac{\tau\ loga}{\kappa}} \hbar(\tau)\ d_q\tau. $$

Proof. $ {}_qS_a[\hbar(\kappa)] = \frac{1}{\kappa(1-q)} \int_0^\infty e_q^{-\frac{\tau\ loga}{\kappa}} \hbar(\tau)\ d_q\tau $

$$ = \frac{1}{\kappa(1-q)} \int_0^M e_q^{-\frac{\tau\ loga}{\kappa}} \hbar(\tau)\ d_q\tau + \frac{1}{\kappa(1-q)} \int_M^\infty e_q^{-\frac{\tau\ loga}{\kappa}} \hbar(\tau)\ d_q\tau. $$

Setting $\tau = C + M$

$$ = \frac{1}{\kappa\ (1-q)} \int_0^M e_q^{-\frac{\tau\ loga}{\kappa}} \hbar(\tau)\ d_q\tau + \frac{1}{\kappa(1-q)} \int_0^\infty e_q^{-\frac{(C+M)\ loga}{\kappa}} \hbar(C+M)\ d_qC $$

$$ = \frac{1}{\kappa\ (1-q)} \int_0^M e_q^{-\frac{\tau\ loga}{\kappa}} \hbar(\tau)\ d_q\tau + \frac{1}{\kappa(1-q)} e_q^{-\frac{M\ loga}{\kappa}} \int_0^\infty e_q^{-\frac{C\ loga}{\kappa}} \hbar(C)\ d_qC $$

$$ = \frac{1}{\kappa\ (1-q)} \int_0^M e_q^{-\frac{\tau\ loga}{\kappa}} \hbar(\tau)\ d_q\tau + e_q^{-\frac{M\ loga}{\kappa}}\ {}_qS_a[\hbar(\tau)] $$

$$ \Rightarrow \left[1 - e_q^{-\frac{M\ loga}{\kappa}}\right]_q S_a[\hbar(\tau)] = \frac{1}{\kappa(1-q)} \int_0^M e_q^{-\frac{\tau\ loga}{\kappa}} \hbar(\tau)\ d_q\tau $$

$$ \Rightarrow_q S_a[\hbar(\tau)] = \frac{\left[1 - e_q^{-\frac{M\ loga}{\kappa}}\right]^{-1}}{\kappa(1-q)} \int_0^M e_q^{-\frac{\tau\ loga}{\kappa}} \hbar(\tau)\ d_q\tau. $$

4. Application

Application 4.1. Solve $\frac{d_q^2\vartheta}{d_q\tau^2} + \vartheta = 0$, under the condition $\vartheta = 1$, $\frac{d_q\vartheta}{d_q\tau} = 0$ when τ =0.

Using the proposed transform, we have

$$ \frac{1}{\kappa^2}\left[(loga)^2_q S_a(\kappa) - loga_q S_a \hbar(0) - \kappa_q S_a \hbar^{/}(0)\right] +_q S_a(\kappa) = 0 $$

or, $\frac{1}{\kappa^2}(loga)^2_q S_a(\kappa) +_q S_a(\kappa) = \frac{1}{(1-q)\kappa^2}$

or, $\left[\frac{1}{\kappa^2}(loga)^2 + 1\right]_q S_a(\kappa) = \frac{1}{\kappa^2(1-q)}$

or, ${}_qS_a(\kappa) = \frac{1}{loga}\left[\frac{loga}{(1-q)\ [(loga)^2 + \kappa^2]}\right]$

or, $\kappa = \frac{cos_q(\tau)}{loga}$.

Application 4.2. Solve $(D_q^2 - 2D_q + 1)\vartheta = 0,\ \ \vartheta = D\vartheta = 1$, when $\tau = 0$.

Using the proposed transform, we have

$$\frac{1}{\kappa^2}\Big((loga)_q^2 S_a(\kappa) - loga_q S_a \hbar(0) - \kappa_q S_a \hbar^{/}(0)\Big) - \frac{2}{\kappa}\Big(loga_q S_a(\kappa) - \hbar(0)\Big)$$

$$+_q S_a(\kappa) = 0$$

or, $\dfrac{1}{\kappa^2}(loga)_q^2 S_a(\kappa) - \dfrac{1}{\kappa^2(1-q)} - \dfrac{1}{\kappa(1-q)loga} - \dfrac{2}{\kappa}loga_q S_a(\kappa) + \dfrac{2}{\kappa(1-q)loga}$

$$+_q S_a(\kappa) = 0$$

or, ${}_q S_a(\kappa)\Big[\dfrac{1}{\kappa^2}(loga)^2 - \dfrac{2}{\kappa}loga + 1\Big] = \dfrac{1}{\kappa^2(1-q)} - \dfrac{1}{(1-q)\ \kappa\ loga}$

or, ${}_q S_a(\kappa) = \dfrac{loga - \kappa}{(1-q)\ loga\ [(loga)^2 - 2\kappa loga + \kappa^2]}$

or, ${}_q S_a(\kappa) = \dfrac{loga - \kappa}{(1-q)\ loga\ (loga - \kappa)^2}$

or, ${}_q S_a(\kappa) = \dfrac{1}{(1-q)\ loga\ (loga - \kappa)}$

or, $\kappa = \dfrac{1}{loga} e_q^{\tau}$.

Application 4.3. We take the first order ODE

$$\frac{d_q\vartheta}{d_q\tau} + \vartheta = 0,\ \ \vartheta(0) = 1.$$

Using the proposed transform, we have

$$\frac{1}{\kappa}\Big(loga_q S_a(\kappa) -_q S_a \hbar(0)\Big) +_q S_a(\kappa) = 0$$

or, $\Big[\dfrac{1}{\kappa}loga + 1\Big]_q S_a(\kappa) = \dfrac{loga}{(1-q)\kappa}$

or, ${}_q S_a(\kappa) = \dfrac{loga}{(1-q)\ (loga + \kappa)}$

or, $\kappa = loga.\ e_q^{-\tau}$.

5. Conclusion

We conclude this work by remarking that we have explored the concept of the q-modified Sumudu transform and established several of its fundamental properties. A set of key theorems has been developed to strengthen the theoretical framework of the transform. Moreover, we have demonstrated its applicability by employing the q-modified Sumudu transform to obtain solutions of ordinary differential equations subject to initial and boundary conditions. These results highlight the effectiveness and potential of the q-modified Sumudu transform as a powerful tool for solving diverse classes of differential equations, as well as showing the importance of the q-analogue [30, 31, 32, 33, 34].

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